

# THE BMO-DIRICHLET PROBLEM FOR ELLIPTIC SYSTEMS IN THE UPPER-HALF SPACE AND QUANTITATIVE CHARACTERIZATIONS OF VMO

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**ABSTRACT.** We prove that for any homogeneous, second order, constant complex coefficient elliptic system  $L$  in  $\mathbb{R}^n$ , the Dirichlet problem in  $\mathbb{R}_+^n$  with boundary data in  $\text{BMO}(\mathbb{R}^{n-1})$  is well-posed in the class of functions  $u$  for which the Littlewood-Paley measure associated with  $u$ , namely  $d\mu_u(x', t) := |\nabla u(x', t)|^2 t \, dx' dt$ , is a Carleson measure in  $\mathbb{R}_+^n$ .

In the process we establish a Fatou type theorem guaranteeing the existence of the pointwise nontangential boundary trace for smooth null-solutions  $u$  of such systems satisfying the said Carleson measure condition. In concert, these results imply that the space  $\text{BMO}(\mathbb{R}^{n-1})$  can be characterized as the collection of nontangential pointwise traces of smooth null-solutions  $u$  to the elliptic system  $L$  with the property that  $\mu_u$  is a Carleson measure in  $\mathbb{R}_+^n$ .

We also establish a regularity result for the BMO-Dirichlet problem in the upper-half space, to the effect that the nontangential pointwise trace on the boundary of  $\mathbb{R}_+^n$  of any given smooth null-solutions  $u$  of  $L$  in  $\mathbb{R}_+^n$  satisfying the above Carleson measure condition actually belongs to Sarason's space  $\text{VMO}(\mathbb{R}^{n-1})$  if and only if  $\mu_u(T(Q))/|Q| \rightarrow 0$  as  $|Q| \rightarrow 0$ , uniformly with respect to the location of the cube  $Q \subset \mathbb{R}^{n-1}$ . In the formulation of the latter vanishing Carleson measure condition,  $T(Q)$  is the Carleson box associated with  $Q$ , and  $|Q|$  denotes the Euclidean volume of  $Q$ .

Moreover, we are able to establish the well-posedness of the Dirichlet problem in  $\mathbb{R}_+^n$  for a system  $L$  as above in the case when the boundary data are prescribed in Morrey-Campanato spaces in  $\mathbb{R}^{n-1}$ . In such a scenario, the solution  $u$  is required to satisfy a vanishing Carleson measure condition of fractional order.

By relying on these well-posedness and regularity results we succeed in producing characterizations of the space VMO as the closure in BMO of classes of smooth functions contained in BMO within which uniform continuity may be suitably quantified (such as the class of smooth functions satisfying a Hölder or Lipschitz condition). This improves on Sarason's classical result describing VMO as the closure in BMO of the space of uniformly continuous functions with bounded mean oscillations.

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## 1. INTRODUCTION AND STATEMENT OF MAIN THEOREMS

Let  $M \in \mathbb{N}$  and consider a second-order, homogeneous,  $M \times M$  system, with constant complex coefficients, written (with the usual convention of summation over repeated indices in place) as

$$Lu := \left( \partial_r (a_{rs}^{\alpha\beta} \partial_s u_\beta) \right)_{1 \leq \alpha \leq M}, \quad (1.1)$$

when acting on a  $\mathcal{C}^2$  vector-valued function  $u = (u_\beta)_{1 \leq \beta \leq M}$  defined in an open subset of  $\mathbb{R}^n$ . Assume that  $L$  is **strongly elliptic** in the sense that there exists  $\kappa_o \in (1, \infty)$  such that

$$\begin{aligned} \operatorname{Re} [a_{rs}^{\alpha\beta} \xi_r \xi_s \overline{\eta_\alpha} \eta_\beta] &\geq \kappa_o |\xi|^2 |\eta|^2 \quad \text{for every} \\ \xi &= (\xi_r)_{1 \leq r \leq n} \in \mathbb{R}^n \quad \text{and} \quad \eta = (\eta_\alpha)_{1 \leq \alpha \leq M} \in \mathbb{C}^M. \end{aligned} \quad (1.2)$$

Examples include scalar operators, such as the Laplacian  $\Delta = \sum_{j=1}^n \partial_j^2$  or, more generally, operators of the form  $\operatorname{div} A \nabla$  with  $A = (a_{rs})_{1 \leq r, s \leq n}$  an  $n \times n$  matrix with complex entries satisfying the ellipticity condition

$$\inf_{\xi \in S^{n-1}} \operatorname{Re} [a_{rs} \xi_r \xi_s] > 0, \quad (1.3)$$

(where  $S^{n-1}$  denotes the unit sphere in  $\mathbb{R}^n$ ), as well as the Lamé system of elasticity

$$Lu := \mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u, \quad u = (u_1, \dots, u_n) \in \mathcal{C}^2. \quad (1.4)$$

Above, the constants  $\lambda, \mu \in \mathbb{R}$  (typically called Lamé moduli), are assumed to satisfy

$$\mu > 0 \quad \text{and} \quad 2\mu + \lambda > 0, \quad (1.5)$$

a condition equivalent to the demand that the Lamé system (1.4) satisfies the Legendre-Hadamard ellipticity condition (1.2). While the Lamé system is symmetric, we stress that the results in this paper require no symmetry for the systems involved.

Returning to the general framework, with every system  $L$  as in (1.1)-(1.2) one may associate a Poisson kernel,  $P^L$ , which is a  $\mathbb{C}^{M \times M}$ -valued function defined in  $\mathbb{R}^{n-1}$  described in detail in Theorem 2.3. This Poisson kernel has played a pivotal role in the treatment of the  $L^p$ -Dirichlet boundary value problem for  $L$  in the upper-half space in [18].

To state our main results, some notation is needed. For a function  $\phi : \mathbb{R}^{n-1} \rightarrow \mathbb{C}$  set

$$\phi_t(x') := t^{1-n} \phi(x'/t), \quad \text{for every } x' \in \mathbb{R}^{n-1} \text{ and every } t > 0. \quad (1.6)$$

In particular,  $P_t^L(x') = t^{1-n} P^L(x'/t)$  for every  $x' \in \mathbb{R}^{n-1}$  and  $t > 0$ . We agree to identify the boundary of the upper-half space

$$\mathbb{R}_+^n := \{x = (x', x_n) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0\} \quad (1.7)$$

with the horizontal hyperplane  $\mathbb{R}^{n-1}$  via  $(x', 0) \equiv x'$  for any  $x' \in \mathbb{R}^{n-1}$ . The origin in  $\mathbb{R}^{n-1}$  is denoted by  $0'$ . Having fixed some background parameter  $\kappa > 0$ , at each point  $x' \in \partial\mathbb{R}_+^n$  we define the conical nontangential approach region with vertex at  $x'$  as

$$\Gamma_\kappa(x') := \{y = (y', t) \in \mathbb{R}_+^n : |x' - y'| < \kappa t\}. \quad (1.8)$$

Whenever meaningful, the nontangential pointwise trace of a continuous vector-valued function  $u$  defined in  $\mathbb{R}_+^n$  is given at each  $x' \in \partial\mathbb{R}_+^n \equiv \mathbb{R}^{n-1}$  by

$$\left(u \Big|_{\partial\mathbb{R}_+^n}^{\text{n.t.}}\right)(x') := \lim_{\Gamma_\kappa(x') \ni y \rightarrow (x', 0)} u(y) \quad \text{for } x' \in \partial\mathbb{R}_+^n \equiv \mathbb{R}^{n-1}. \quad (1.9)$$

For each positive integer  $k$  denote by  $\mathcal{L}^k$  the  $k$ -dimensional Lebesgue measure in  $\mathbb{R}^k$ . A Borel measure  $\mu$  in  $\mathbb{R}_+^n$  is said to be a Carleson measure in  $\mathbb{R}_+^n$  provided

$$\|\mu\|_{\mathcal{C}(\mathbb{R}_+^n)} := \sup_{Q \subset \mathbb{R}^{n-1}} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q d\mu(x', t) < \infty, \quad (1.10)$$

where the supremum runs over all cubes  $Q$  in  $\mathbb{R}^{n-1}$ . Here and elsewhere in the paper, by a cube  $Q$  in  $\mathbb{R}^{n-1}$  we shall understand a cube with sides parallel to the coordinate axes, and its side-length will be denoted by  $\ell(Q)$ . Also, the  $\mathcal{L}^{n-1}$  measure of  $Q$  is denoted by  $|Q|$  and if  $\lambda > 0$  then  $\lambda Q$  denotes the cube concentric with  $Q$  whose side-length is  $\lambda \ell(Q)$ . Call a Borel measure  $\mu$  in  $\mathbb{R}_+^n$  a vanishing Carleson measure whenever  $\mu$  is a Carleson measure to begin with and, in addition,

$$\lim_{r \rightarrow 0^+} \left( \sup_{Q \subset \mathbb{R}^{n-1}, \ell(Q) \leq r} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q d\mu(x', t) \right) = 0. \quad (1.11)$$

Next, the Littlewood-Paley measure associated with a continuously differentiable function  $u$  in  $\mathbb{R}_+^n$  is  $|\nabla u(x', t)|^2 t dx' dt$ , and we set

$$\|u\|_{**} := \sup_{Q \subset \mathbb{R}^{n-1}} \left( \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\nabla u(x', t)|^2 t dx' dt \right)^{\frac{1}{2}}. \quad (1.12)$$

In particular, for a continuously differentiable function  $u$  in  $\mathbb{R}_+^n$  we have

$$\|u\|_{**} < \infty \iff |\nabla u(x', t)|^2 t dx' dt \text{ is a Carleson measure in } \mathbb{R}_+^n. \quad (1.13)$$

We next introduce  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ , the John-Nirenberg space of vector-valued functions of bounded mean oscillations in  $\mathbb{R}^{n-1}$ , as the collection of  $\mathbb{C}^M$ -valued functions  $f = (f_\alpha)_{1 \leq \alpha \leq M}$  with components in  $L_{\text{loc}}^1(\mathbb{R}^{n-1})$  satisfying

$$\|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} := \sup_{Q \subset \mathbb{R}^{n-1}} \oint_Q |f(x') - f_Q| dx' < \infty. \quad (1.14)$$

Above, for every cube  $Q$  in  $\mathbb{R}^{n-1}$  and every function  $h \in L_{\text{loc}}^1(\mathbb{R}^{n-1}, \mathbb{C}^M)$  we have abbreviated

$$h_Q := \oint_Q h(x') dx' := \frac{1}{|Q|} \int_Q h(x') dx' \in \mathbb{C}^M, \quad (1.15)$$

where the last integration is performed componentwise. To lighten notation, when  $M = 1$  we simply write  $\text{BMO}(\mathbb{R}^{n-1})$  in place of  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C})$ . Clearly, for every  $f \in L_{\text{loc}}^1(\mathbb{R}^{n-1}, \mathbb{C}^M)$  we have

$$\begin{aligned} \|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} &= \|f + C\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)}, & \forall C \in \mathbb{C}^M, \\ \|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} &= \|\tau_{z'} f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)}, & \forall z' \in \mathbb{R}^{n-1}, \\ \|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} &= \|\delta_\lambda f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)}, & \forall \lambda \in (0, \infty), \end{aligned} \quad (1.16)$$

where  $\tau_{z'}$  is the operator of translation by  $z'$ , i.e.,  $(\tau_{z'}f)(x') := f(x' + z')$  for every  $x' \in \mathbb{R}^{n-1}$ , and  $\delta_\lambda$  is the operator of dilation by  $\lambda$ , i.e.,  $(\delta_\lambda f)(x') := f(\lambda x')$  for every  $x' \in \mathbb{R}^{n-1}$ .

We wish to note here that  $\|\cdot\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)}$  is only a seminorm, since for every function  $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  we have

$$\|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} = 0 \iff f \text{ is constant } \mathcal{L}^{n-1}\text{-a.e. in } \mathbb{R}^{n-1} \text{ (in } \mathbb{C}^M). \quad (1.17)$$

Occasionally, we find it useful to mod out its null-space, in order to render the resulting quotient space Banach. Specifically, for two functions  $f, g \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  we say that  $f \sim g$  provided  $f - g$  is constant  $\mathcal{L}^{n-1}$ -a.e. in  $\mathbb{R}^{n-1}$ . This is an equivalence relation on  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  and for each  $f \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  we set

$$[f] := \{g \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) : f \sim g\}, \quad \text{and} \quad (1.18)$$

$$\|[f]\|_{\widetilde{\text{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M)} := \|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)}. \quad (1.19)$$

Then  $\|[\cdot]\|_{\widetilde{\text{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M)}$  becomes a genuine norm on the quotient space

$$\widetilde{\text{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M) := \{[f] : f \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)\}. \quad (1.20)$$

In fact, when equipped with the norm (1.19), the space (1.20) is complete (hence Banach).

Moving on, the Sarason space of  $\mathbb{C}^M$ -valued functions of vanishing mean oscillations in  $\mathbb{R}^{n-1}$  is defined by

$$\text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) := \left\{ f \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) : \lim_{r \rightarrow 0^+} \left( \sup_{Q \subset \mathbb{R}^{n-1}, \ell(Q) \leq r} \int_Q |f(x') - f_Q| dx' \right) = 0 \right\}. \quad (1.21)$$

The space  $\text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  turns out to be a closed subspace of  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . In fact, if  $\text{UC}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  stands for the space of  $\mathbb{C}^M$ -valued uniformly continuous functions in  $\mathbb{R}^{n-1}$ , a well-known result of Sarason [27, Theorem 1, p. 392] implies that

$$\begin{aligned} & \text{a given function } f \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) \text{ actually belongs} \\ & \text{to the space } \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) \text{ if and only if there exists} \\ & \text{a sequence } \{f_j\}_{j \in \mathbb{N}} \subset \text{UC}(\mathbb{R}^{n-1}, \mathbb{C}^M) \cap \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) \\ & \text{such that } \|f - f_j\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \longrightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned} \quad (1.22)$$

We shall refer to this simply by saying that Sarason's VMO space is the closure of  $\text{UC} \cap \text{BMO}$  in the space  $\text{BMO}$ . Another characterization of  $\text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  due to Sarason (cf. [27, Theorem 1, p. 392]) is as follows:

$$\begin{aligned} & \text{a given function } f \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) \text{ actually} \\ & \text{belongs to the space } \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) \text{ if and} \\ & \text{only if } \lim_{\mathbb{R}^{n-1} \ni z' \rightarrow 0'} \|\tau_{z'} f - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} = 0. \end{aligned} \quad (1.23)$$

We are now ready to state our first main result. This concerns the well-posedness of the BMO-Dirichlet problem in the upper-half space for systems  $L$  as in (1.1)-(1.2). The existence of a unique solution is established in the class of functions  $u$  satisfying a Carleson measure condition (expressed in terms of the finiteness of (1.12)). The formulation of our theorem emphasizes the fact that this contains as a “sub-problem” the VMO-Dirichlet problem for  $L$  in  $\mathbb{R}_+^n$  (in which scenario  $u$  satisfies a vanishing Carleson measure condition).

**Theorem 1.1.** *Let  $L$  be an  $M \times M$  elliptic constant complex coefficient system as in (1.1)-(1.2). Then the BMO-Dirichlet boundary value problem for  $L$  in  $\mathbb{R}_+^n$ , namely*

$$\begin{cases} u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M), \\ Lu = 0 \text{ in } \mathbb{R}_+^n, \\ |\nabla u(x', t)|^2 t dx' dt \text{ is a Carleson measure in } \mathbb{R}_+^n, \\ u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} = f \text{ a.e. in } \mathbb{R}^{n-1}, f \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M), \end{cases} \quad (1.24)$$

has a unique solution. Moreover, this unique solution satisfies the following additional properties:

(i) With  $P^L$  denoting the Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.3, one has the Poisson integral representation formula

$$u(x', t) = (P_t^L * f)(x'), \quad \forall (x', t) \in \mathbb{R}_+^n. \quad (1.25)$$

(ii) There exists a constant  $C = C(n, L) \in (1, \infty)$  with the property that the solution  $u$  of the Dirichlet problem (1.24) satisfies the two-sided estimate

$$C^{-1} \|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq \|u\|_{**} \leq C \|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)}. \quad (1.26)$$

That is, the size of the solution is comparable to the size of the boundary datum.

(iii) For each  $\varepsilon > 0$  the function  $u(\cdot, \varepsilon)$  belongs to  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  and there exists a constant  $C = C(n, L) \in (0, \infty)$  independent of  $u$  with the property that the following uniform BMO estimate holds:

$$\sup_{\varepsilon > 0} \|u(\cdot, \varepsilon)\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq C \|u\|_{**}. \quad (1.27)$$

Moreover,

$$\lim_{\varepsilon \rightarrow 0^+} \|u(\cdot, \varepsilon) - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} = 0 \iff \begin{cases} |\nabla u(x', t)|^2 t dx' dt \text{ is a vanishing} \\ \text{Carleson measure in } \mathbb{R}_+^n. \end{cases} \quad (1.28)$$

That is,  $u$  satisfies a vanishing Carleson measure condition in  $\mathbb{R}_+^n$  if and only if  $u$  converges to its boundary datum vertically in  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ .

(iv) The following regularity results hold:

$$f \in \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) \iff \begin{cases} |\nabla u(x', t)|^2 t dx' dt \text{ is a vanishing} \\ \text{Carleson measure in } \mathbb{R}_+^n \end{cases} \quad (1.29)$$

$$\iff \lim_{\mathbb{R}_+^n \ni z \rightarrow 0} \|\tau_z u - u\|_{**} = 0, \quad (1.30)$$

where  $(\tau_z u)(x) := u(x + z)$  for each  $x, z \in \mathbb{R}_+^n$ .

As a consequence, the VMO-Dirichlet boundary value problem for  $L$  in  $\mathbb{R}_+^n$ , i.e.,

$$\begin{cases} u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M), \\ Lu = 0 \text{ in } \mathbb{R}_+^n, \\ |\nabla u(x', t)|^2 t dx' dt \text{ is a vanishing Carleson measure in } \mathbb{R}_+^n, \\ u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} = f \text{ a.e. in } \mathbb{R}^{n-1}, f \in \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M), \end{cases} \quad (1.31)$$

is well-posed. Moreover, its unique solution is given by (1.25), satisfies (1.26)-(1.27), and

$$\lim_{\varepsilon \rightarrow 0^+} \|u(\cdot, \varepsilon) - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} = 0. \quad (1.32)$$

It is reassuring to remark that replacing the boundary datum  $f$  by  $f + C$  where  $C \in \mathbb{C}^M$  in (1.24) changes the solution  $u$  into  $u + C$  (given that convolution with the Poisson kernel reproduces constants from  $\mathbb{C}^M$ ; cf. (2.36)). As such, the  $\widetilde{\text{BMO}}$ -Dirichlet problem for  $L$  in  $\mathbb{R}_+^n$  is also well-posed, if uniqueness of the solution is now understood modulo constants from  $\mathbb{C}^M$ .

As regards the right-pointing implication in (1.29), for suitable dense subspaces of VMO we are able to precisely quantify the rate at which the Carleson measure  $|\nabla u(x', t)|^2 t dx' dt$  vanishes in  $\mathbb{R}_+^n$ . For example, with  $\dot{\mathcal{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)$  denoting the homogeneous Hölder space of order  $\eta$  of  $\mathbb{C}^M$ -valued functions defined in  $\mathbb{R}^{n-1}$ , it follows from (3.9) in Proposition 3.1 (cf. also (2.19)) that

$$\text{if } f \in \dot{\mathcal{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M) \text{ with } \eta \in (0, 1) \text{ and } u \text{ is as in (1.25), then} \quad (1.33)$$

$$\sup_{Q \subset \mathbb{R}^{n-1}, \ell(Q) \leq r} \left( \int_0^{\ell(Q)} \int_Q |\nabla u(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} = O(r^\eta) \text{ as } r \rightarrow 0^+,$$

where the multiplicative constant implicit in the big- $O$  condition above depends only on  $n, L, \eta$ , and  $\|f\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)}$ . The relevance of this result stems from the fact that, for each  $\eta \in (0, 1)$ , the collection of functions from  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  which also belong to  $\dot{\mathcal{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)$  make up a dense subspace of  $\text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . The latter density result constitutes one of the main results in this paper, and is formally stated in Theorem 1.4, along with a number of variants and generalizations. Let us also point out here that the decay rate in (1.33) is in agreement with the format of the well-posedness result proved later in Theorem 1.9 (in view of (1.73) and (1.69)).

The proof of Theorem 1.1 relies on a quantitative Fatou type theorem, which includes a Poisson integral representation formula along with a characterization of BMO in terms of the traces of solutions to elliptic systems. This is stated next as Theorem 1.2. Among other things, the said theorem shows that the conditions stipulated in the first three lines of (1.24) imply that the pointwise nontangential limit considered in the fourth line of (1.24) is always meaningful, and that the boundary datum should necessarily be selected from the space BMO. It also highlights the fact that it is natural to seek a solution of the BMO Dirichlet problem by taking the convolution of the boundary datum with the Poisson kernel of  $L$  in the upper-half space. Finally, Theorem 1.2 is the key ingredient in the proof of uniqueness for the BMO-Dirichlet boundary value problem formulated in (1.24).

**Theorem 1.2.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2) and consider  $P^L$ , the associated Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.3. Then there exists a constant  $C = C(L, n) \in (1, \infty)$  with the property that*

$$\left. \begin{array}{l} u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M) \\ Lu = 0 \text{ in } \mathbb{R}_+^n \\ \text{and } \|u\|_{**} < \infty \end{array} \right\} \implies \left\{ \begin{array}{l} u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} \text{ exists a.e. in } \mathbb{R}^{n-1}, \text{ lies in } \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M), \\ u(x', t) = \left( P_t^L * (u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}}) \right)(x') \text{ for all } (x', t) \in \mathbb{R}_+^n, \\ \text{and } C^{-1} \|u\|_{**} \leq \|u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}}\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq C \|u\|_{**}. \end{array} \right. \quad (1.34)$$

In fact, the following characterization of  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ , adapted to the system  $L$ , holds:

$$\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) = \left\{ u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} : u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M), Lu = 0 \text{ in } \mathbb{R}_+^n, \|u\|_{**} < \infty \right\}. \quad (1.35)$$

Moreover,

$$\text{LMO}(\mathbb{R}_+^n) := \left\{ u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M) : Lu = 0 \text{ in } \mathbb{R}_+^n, \|u\|_{**} < \infty \right\} \quad (1.36)$$



is a linear space on which  $\|\cdot\|_{**}$  is a seminorm with null-space  $\mathbb{C}^M$ , the quotient space  $\text{LMO}(\mathbb{R}_+^n)/\mathbb{C}^M$  becomes complete (hence Banach) when equipped with  $\|\cdot\|_{**}$ , and the non-tangential pointwise trace operator acting on equivalence classes in the context

$$\text{LMO}(\mathbb{R}_+^n)/\mathbb{C}^M \ni [u] \longmapsto [u]_{\partial\mathbb{R}_+^n}^{\text{n.t.}} \in \widetilde{\text{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M) \quad (1.37)$$

is a well-defined linear isomorphism between Banach spaces, where  $[u]$  in (1.37) denotes the equivalence class of  $u$  in  $\text{LMO}(\mathbb{R}_+^n)/\mathbb{C}^M$  and  $[u]_{\partial\mathbb{R}_+^n}^{\text{n.t.}}$  is interpreted as in (1.18).

There is a counterpart of the Fatou type result stated as Theorem 1.2 emphasizing the space VMO in place of BMO. Specifically, we prove the following theorem.

**Theorem 1.3.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2) and consider  $P^L$ , the associated Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.3. Then for any function*

$$u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M) \text{ satisfying } Lu = 0 \text{ in } \mathbb{R}_+^n \text{ and } \|u\|_{**} < \infty \quad (1.38)$$

one has

$$\left. \begin{array}{l} |\nabla u(x', t)|^2 t \, dx' dt \text{ is a vanishing} \\ \text{Carleson measure in } \mathbb{R}_+^n \end{array} \right\} \implies u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} \in \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M). \quad (1.39)$$

Furthermore, the following characterization of the space  $\text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ , adapted to the system  $L$ , holds:

$$\begin{aligned} \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) = \left\{ u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} : u \in \text{LMO}(\mathbb{R}_+^n, \mathbb{C}^M) \text{ and } |\nabla u(x', t)|^2 t \, dx' dt \right. \\ \left. \text{is a vanishing Carleson measure in } \mathbb{R}_+^n \right\}. \end{aligned} \quad (1.40)$$

In our next result we shall revisit the issue of describing VMO as the closure within BMO of a subspace of functions whose pointwise oscillations vanish as the scale decreases to zero. One such description is contained in (1.22). However, for a variety of purposes (such as the proof of the result recorded in Corollary 1.7 below), the fact that the condition of uniform continuity is of a purely qualitative nature renders the space UC difficult to work with. As such, it is very desirable to replace it –in the context of Sarason’s density result recorded in (1.22)– by smaller subspaces within which uniform continuity may be suitably quantified. This issue is addressed in Theorem 1.4 below. As a preamble, we introduce some notation. Pick a modulus of continuity, i.e., a function

$$\Upsilon : [0, \infty) \rightarrow [0, \infty] \text{ nondecreasing and such that } \lim_{s \rightarrow 0^+} \Upsilon(s) = 0. \quad (1.41)$$

Given  $m \in \mathbb{N}$ , consider the space

$$\mathcal{C}^\Upsilon(\mathbb{R}^m) := \{f : \mathbb{R}^m \rightarrow \mathbb{C} : \text{there exists } C \in (0, \infty) \text{ such that}$$

$$|f(a) - f(b)| \leq C\Upsilon(|a - b|) \text{ for all } a, b \in \mathbb{R}^m\} \quad (1.42)$$

and define  $\|f\|_{\mathcal{C}^\Upsilon(\mathbb{R}^m)}$  to be the smallest constant  $C$  intervening above. In the sequel, the space of  $\mathbb{C}^M$ -valued functions with components in  $\mathcal{C}^\Upsilon(\mathbb{R}^m)$  will be denoted by  $\mathcal{C}^\Upsilon(\mathbb{R}^m, \mathbb{C}^M)$ . Clearly,  $\mathcal{C}^\Upsilon(\mathbb{R}^m) \subseteq \text{UC}(\mathbb{R}^m)$  and, in fact,

$$\text{UC}(\mathbb{R}^m) = \bigcup_{\substack{\Upsilon \text{ modulus of} \\ \text{continuity}}} \mathcal{C}^\Upsilon(\mathbb{R}^m). \quad (1.43)$$

Examples of interest are obtained by taking  $\eta \in (0, 1]$  and defining  $\Upsilon_\eta(s) := s^\eta$  for every  $s \geq 0$ . Then the space  $\mathcal{C}^{\Upsilon_\eta}(\mathbb{R}^m)$  becomes precisely  $\mathcal{C}^\eta(\mathbb{R}^m)$ , the space of functions satisfying

a homogeneous Hölder condition of order  $\eta$  in  $\mathbb{R}^m$  in the case when  $\eta \in (0, 1)$ , and becomes  $\text{Lip}(\mathbb{R}^m)$ , the space of Lipschitz functions in  $\mathbb{R}^m$ , in the case when  $\eta = 1$ .

Here is the theorem advertised earlier, which may be regarded as a quantitative description of VMO, improving on Sarason's classical result (1.22).

**Theorem 1.4.** *Consider the function  $\Upsilon_{\#} : [0, \infty) \rightarrow [0, \infty)$  given at each  $s \geq 0$  by*

$$\Upsilon_{\#}(s) := \min\{1, s\} + \max\{0, \ln s\} = \begin{cases} s, & \text{if } s \leq 1, \\ 1 + \ln s, & \text{if } s > 1. \end{cases} \quad (1.44)$$

*Then for every modulus of continuity  $\Upsilon$  with the property that  $\Upsilon_{\#} \leq C\Upsilon$  on  $[0, \infty)$  for some finite constant  $C > 0$ , the following density result holds for each  $n \in \mathbb{N}$ :*

$$\begin{aligned} &\text{for every function } f \in \text{VMO}(\mathbb{R}^n) \text{ there exists a se-} \\ &\text{quence } \{f_j\}_{j \in \mathbb{N}} \subset \mathcal{C}^{\Upsilon}(\mathbb{R}^n) \cap \mathcal{C}^{\infty}(\mathbb{R}^n) \cap \text{BMO}(\mathbb{R}^n) \\ &\text{such that } \|f - f_j\|_{\text{BMO}(\mathbb{R}^n)} \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned} \quad (1.45)$$

*In short,  $\mathcal{C}^{\Upsilon}(\mathbb{R}^n) \cap \mathcal{C}^{\infty}(\mathbb{R}^n) \cap \text{BMO}(\mathbb{R}^n)$  is dense in  $\text{VMO}(\mathbb{R}^n)$ . In fact,*

$$\begin{aligned} &\text{the smaller space, consisting of } f \in \mathcal{C}^{\Upsilon}(\mathbb{R}^n) \cap \mathcal{C}^{\infty}(\mathbb{R}^n) \cap \text{BMO}(\mathbb{R}^n) \\ &\text{such that } \partial^{\alpha} f \in \mathcal{C}^{\Upsilon}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n) \text{ for every } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \geq 1, \text{ is} \\ &\text{also dense in } \text{VMO}(\mathbb{R}^n). \end{aligned} \quad (1.46)$$

Several density results, of independent interest, are obtained by specializing Theorem 1.4 to moduli of continuity of the form  $\Upsilon_{\eta}(s) := s^{\eta}$  for  $s \geq 0$ , with  $\eta \in (0, 1]$  (simply by observing that there exists some finite constant  $C_{\eta} > 0$  with the property that  $\Upsilon_{\#} \leq C_{\eta} \Upsilon_{\eta}$  on  $[0, \infty)$ ). To state these, recall that the inhomogeneous Hölder space of order  $\eta \in (0, 1)$  in  $\mathbb{R}^n$  is defined as

$$\mathcal{C}^{\eta}(\mathbb{R}^n) := \dot{\mathcal{C}}^{\eta}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n). \quad (1.47)$$

**Corollary 1.5.** *For each  $\eta \in (0, 1)$ ,*

$$\begin{aligned} &\text{the space consisting of } f \in \dot{\mathcal{C}}^{\eta}(\mathbb{R}^n) \cap \mathcal{C}^{\infty}(\mathbb{R}^n) \cap \text{BMO}(\mathbb{R}^n) \text{ such that} \\ &\partial^{\alpha} f \in \mathcal{C}^{\eta}(\mathbb{R}^n) \text{ for every } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \geq 1 \text{ is dense in } \text{VMO}(\mathbb{R}^n). \end{aligned} \quad (1.48)$$

*In particular, for each  $\eta \in (0, 1)$ ,*

$$\begin{aligned} &\dot{\mathcal{C}}^{\eta}(\mathbb{R}^n) \cap \mathcal{C}^{\infty}(\mathbb{R}^n) \cap \text{BMO}(\mathbb{R}^n) \\ &\text{is a dense subspace of } \text{VMO}(\mathbb{R}^n). \end{aligned} \quad (1.49)$$

*Moreover,*

$$\begin{aligned} &\text{the space consisting of functions } f \in \text{Lip}(\mathbb{R}^n) \cap \mathcal{C}^{\infty}(\mathbb{R}^n) \cap \text{BMO}(\mathbb{R}^n) \text{ such} \\ &\text{that } \partial^{\alpha} f \in \text{Lip}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n) \text{ for every } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \geq 1 \text{ is dense in} \\ &\text{VMO}(\mathbb{R}^n). \end{aligned} \quad (1.50)$$

*In particular,*

$$\begin{aligned} &\text{Lip}(\mathbb{R}^n) \cap \mathcal{C}^{\infty}(\mathbb{R}^n) \cap \text{BMO}(\mathbb{R}^n) \\ &\text{is a dense subspace of } \text{VMO}(\mathbb{R}^n). \end{aligned} \quad (1.51)$$

An interesting feature of Theorem 1.4 is that even though the conclusions are of a purely real-variable nature, its proof makes essential use the PDE-rooted results established earlier (such as the well-posedness of the BMO-Dirichlet problem for, say, the Laplacian in  $\mathbb{R}_+^n$ ). See §5 for details. Theorem 1.4 should be contrasted with the following negative result.

**Theorem 1.6.** *The space  $\text{UC}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$  is not dense in  $\text{VMO}(\mathbb{R}^n)$ .*



In the context of the main density result presented in Theorem 1.4, the function  $\Upsilon_{\#}$  defined in (1.44) exhibits an optimal behavior both at small and large scales, which cannot be improved, in the following precise sense: *If  $\Upsilon$  is a modulus of continuity with the property that*

$$\text{either } \Upsilon(s)/s = o(1) \text{ as } s \rightarrow 0^+, \text{ or } \Upsilon(s) = O(1) \text{ as } s \rightarrow \infty, \quad (1.52)$$

*then*

$$\mathcal{C}^{\Upsilon}(\mathbb{R}^n) \cap \text{BMO}(\mathbb{R}^n) \text{ is not dense in } \text{VMO}(\mathbb{R}^n). \quad (1.53)$$

Indeed, (1.53) is clear when the first eventuality in (1.52) materializes since the space  $\mathcal{C}^{\Upsilon}(\mathbb{R}^n)$  reduces to just constants in this case. Also, in the scenario when the second possibility in (1.52) takes place,  $\mathcal{C}^{\Upsilon}(\mathbb{R}^n)$  becomes a subspace of  $\text{UC}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ , in which case the desired conclusion follows from Theorem 1.6.

It is also instructive to note here that the quantitative characterizations of the Sarason VMO space provided in Theorem 1.4 yield a simple, and conceptually natural, proof of the fact that the Riesz transforms in  $\mathbb{R}^n$ , defined for each index  $j \in \{1, \dots, n\}$  as

$$(R_j f)(x) := \lim_{\varepsilon \rightarrow 0^+} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{y \in \mathbb{R}^n \setminus B(x, \varepsilon)} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) dy, \quad x \in \mathbb{R}^n, \quad (1.54)$$

are bounded from  $\text{VMO}(\mathbb{R}^n)$  into itself. In fact, a more general result of this nature is true. Recall that a linear and bounded mapping  $T : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)$  is said to be a Calderón-Zygmund operator provided  $T$  extends to a bounded operator on  $L^2(\mathbb{R}^n)$  and there exist  $C \in (0, \infty)$  and  $\gamma \in (0, 1)$  such that its Schwartz kernel  $K(\cdot, \cdot)$  satisfies, for every  $x, y \in \mathbb{R}^n$  with  $x \neq y$ ,

$$\begin{aligned} K &\in L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^n \setminus \text{diag}), \quad |K(x, y)| \leq C|x - y|^{-n}, \\ |K(x, y) - K(z, y)| &\leq C|x - z|^{\gamma}|x - y|^{-n-\gamma}, \quad z \in \mathbb{R}^n, \quad |x - z| \leq \frac{1}{2}|x - y|, \\ |K(x, y) - K(x, z)| &\leq C|y - z|^{\gamma}|x - y|^{-n-\gamma}, \quad z \in \mathbb{R}^n, \quad |y - z| \leq \frac{1}{2}|x - y|. \end{aligned} \quad (1.55)$$

**Corollary 1.7.** *Any Calderón-Zygmund operator  $T$  in  $\mathbb{R}^n$  with the property that  $T(1) = 0$  is bounded from  $\text{VMO}(\mathbb{R}^n)$  into itself.*

Indeed, by [24, Corollaire, p. 239],  $T$  maps  $\text{BMO}(\mathbb{R}^n)$  boundedly into itself, while by [24, Théorème 1, p. 328], given any  $\eta \in (0, 1)$ , the operator  $T$  maps  $\mathcal{C}^{\eta}(\mathbb{R}^n)$  boundedly into itself. As such,  $T$  maps the closure of  $\mathcal{C}^{\eta}(\mathbb{R}^n) \cap \text{BMO}(\mathbb{R}^n)$ , taken in the space  $\text{BMO}(\mathbb{R}^n)$  and identified to be  $\text{VMO}(\mathbb{R}^n)$  in (1.49), boundedly into itself. The argument which proves Corollary 1.7 is indicative of a more general principle at play here, to the effect that

$$\begin{aligned} &\text{any linear operator which is bounded both on BMO and on} \\ &\text{a (homogeneous) Hölder space is also bounded on VMO.} \end{aligned} \quad (1.56)$$

In relation to (1.56), it is also worth pointing out that the class of operators which are simultaneously bounded on BMO as well as on some common (homogeneous) Hölder space is considerably larger than that of the Calderón-Zygmund operators considered in Corollary 1.7 since, as opposed to the latter, the former is stable under composition (hence, in particular, constitutes an algebra).

The proof of Theorem 1.4 (stated with  $n - 1$  in place of  $n$ ) relies on the fact that, given any  $f \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ , we have (as seen from (1.25) and (1.28)-(1.29))

$$P_t^L * f \longrightarrow f \text{ in } \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) \text{ as } t \rightarrow 0^+ \iff f \in \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M), \quad (1.57)$$

for some (or any)  $M \times M$  elliptic system  $L$  with constant complex coefficients as in (1.1)-(1.2). A posteriori, once the density result in Theorem 1.4 has been established, we can considerably enlarge the class of approximations to the identity for which a result as in (1.57) holds, as described below.

**Theorem 1.8.** *Suppose  $\varphi : \mathbb{R}^n \rightarrow \mathbb{C}^{M \times M}$  has the property that there exist  $C \in (0, \infty)$  and  $\varepsilon \in (0, 1]$  such that*

$$|\varphi(x)| \leq C(1 + |x|)^{-n-\varepsilon} \quad \text{for every } x \in \mathbb{R}^n \setminus \{0\}, \quad (1.58)$$

and

$$|\varphi(x+h) - \varphi(x)| \leq \frac{C|h|^\varepsilon}{|x|^{n+\varepsilon}} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}, \quad h \in \mathbb{R}^n, \quad |h| < |x|/2. \quad (1.59)$$

In addition, assume

$$\int_{\mathbb{R}^n} \varphi(x) dx = I_{M \times M} \quad (1.60)$$

(the  $M \times M$  identity matrix). Then

$$\begin{aligned} &\text{for each } f \in \text{VMO}(\mathbb{R}^n, \mathbb{C}^M) \text{ there holds} \\ &\varphi_t * f \longrightarrow f \text{ in } \text{BMO}(\mathbb{R}^n, \mathbb{C}^M), \text{ as } t \rightarrow 0^+, \end{aligned} \quad (1.61)$$

where, in the present context,  $\varphi_t(x) := t^{-n}\varphi(x/t)$  for each  $x \in \mathbb{R}^n$  and each  $t > 0$ .

As a consequence, given  $\varphi \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{C}^{M \times M})$  such that (1.60) holds and such that there exists  $C \in (0, \infty)$  for which

$$|\varphi(x)| + |(\nabla \varphi)(x)| \leq C(1 + |x|)^{-n-1} \quad \text{for every } x \in \mathbb{R}^n, \quad (1.62)$$

one has the following real-variable characterization of the membership to VMO:

$$\begin{aligned} &\text{for every function } f \in \text{BMO}(\mathbb{R}^n, \mathbb{C}^M) \text{ there holds} \\ &\varphi_t * f \rightarrow f \text{ in } \text{BMO}(\mathbb{R}^n, \mathbb{C}^M) \text{ as } t \rightarrow 0^+ \iff f \in \text{VMO}(\mathbb{R}^n, \mathbb{C}^M). \end{aligned} \quad (1.63)$$

This paper is part of a larger program aimed at treating Dirichlet boundary value problems for  $M \times M$  systems with constant complex coefficients as in (1.1)-(1.2) in the upper half-space  $\mathbb{R}_+^n$  with boundary datum in various function spaces on  $\mathbb{R}^{n-1}$ . The space BMO, presently considered, lies at the cross-roads of several fundamental scales of function spaces in analysis. For one thing,  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  may be regarded as a natural (right-most) end-point of the Lebesgue scale  $L^p(\mathbb{R}^{n-1}, \mathbb{C}^M)$  with  $p \in (1, \infty)$ . The Dirichlet boundary value problem for elliptic systems  $L$  as in (1.1)-(1.2) in the upper-half space with data from the latter scale of spaces has been recently treated in [18], where the size of the solution  $u : \mathbb{R}_+^n \rightarrow \mathbb{C}^M$  is measured using the nontangential maximal operator defined as

$$(\mathcal{N}u)(x') := (\mathcal{N}_\kappa u)(x') := \sup \{|u(y)| : y \in \Gamma_\kappa(x')\}, \quad \forall x' \in \mathbb{R}^{n-1}. \quad (1.64)$$

In this endeavor, the crux of the matter is the pointwise inequality (see (2.40))

$$\begin{aligned} &(\mathcal{N}u)(x') \leq C(\mathcal{M}f)(x') \text{ at each point } x' \in \mathbb{R}^{n-1} \\ &\text{if } u(x', t) := (P_t^L * f)(x') \text{ for every } (x', t) \in \mathbb{R}_+^n \\ &\text{and for some function } f \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n} dx'\right)^M, \end{aligned} \quad (1.65)$$

where  $\mathcal{M}$  is the Hardy-Littlewood maximal operator on  $\mathbb{R}^{n-1}$  (cf. (2.4)).

In fact, estimate (1.65) permitted the treatment in [18] of a much larger variety of function lattice spaces. Indeed, one of the main results established in [18] is that the boundedness of the Hardy-Littlewood maximal operator on a Köthe function space  $\mathbb{X}$  and on its Köthe dual  $\mathbb{X}'$  (both considered in  $\mathbb{R}^{n-1}$ ) is actually equivalent to the well-posedness of the  $\mathbb{X}$ -Dirichlet and  $\mathbb{X}'$ -Dirichlet problems in  $\mathbb{R}_+^n$  in the class of all second-order, homogeneous, elliptic systems, with constant complex coefficients. As a consequence, in [18] the Dirichlet problem for such systems has been shown to be well-posed for boundary data in Lebesgue spaces, variable exponent Lebesgue spaces, Lorentz spaces, Zygmund spaces, as well as their weighted versions with weights in the Muckenhoupt class.

This being said, the John-Nirenberg space  $\text{BMO}(\mathbb{R}^{n-1})$  is not a lattice space (in the sense that a nonnegative measurable function with a pointwise majorant in BMO does not necessarily belong to BMO), so a fresh look at the corresponding Dirichlet problem is warranted. In particular, the nature of the space of solutions (which should be suitably tailored to the specific space of boundary data) now involves a Carleson measure condition in place of the nontangential maximal operator (1.64) which has been extensively used in [18].

Another point of view places the John-Nirenberg space  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  as a (left-most) endpoint for the scale of homogeneous Hölder spaces  $\mathcal{C}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)$  with  $\eta \in (0, 1)$  (for pertinent definitions and basic properties regarding this scale see the discussion in the first part of §2). Bearing this in mind, it is possible to formulate (a significant portion of) Theorem 1.1 in a manner that reflects the aforementioned feature of BMO. To elaborate on this idea, given  $\eta \in [0, 1)$  and  $p \in [1, \infty)$ , for every  $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  define

$$\|f\|_*^{(\eta, p)} := \sup_{Q \subset \mathbb{R}^{n-1}} \ell(Q)^{-\eta} \left( \int_Q |f(x') - f_Q|^p dx' \right)^{\frac{1}{p}}, \quad (1.66)$$

and introduce the Morrey-Campanato space (which may be regarded as a fractional BMO space,  $L^p$ -based, of order  $\eta$ ) by setting

$$\mathcal{E}^{\eta, p}(\mathbb{R}^{n-1}, \mathbb{C}^M) := \{f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M) : \|f\|_*^{(\eta, p)} < \infty\}. \quad (1.67)$$

By the John-Nirenberg inequality it follows that, corresponding to the end-point case  $\eta = 0$  we have

$$\mathcal{E}^{0, p}(\mathbb{R}^{n-1}, \mathbb{C}^M) = \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M), \quad (1.68)$$

and it is clear from definitions that, in the regime  $\eta > 0$ , the vanishing mean oscillation condition (1.21) holds (this time, at a precisely quantified rate of decay) for every function  $f \in \mathcal{E}^{\eta, p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . Going further, for every  $u \in \mathcal{C}^1(\mathbb{R}_+^n, \mathbb{C}^M)$  set

$$\|u\|_{**}^{(\eta, p)} := \sup_{Q \subset \mathbb{R}^{n-1}} \ell(Q)^{-\eta} \left( \int_Q \left( \int_0^{\ell(Q)} |\nabla u(x', t)|^2 t dt \right)^{\frac{p}{2}} dx' \right)^{\frac{1}{p}}. \quad (1.69)$$

The finiteness demand  $\|u\|_{**}^{(\eta, p)} < \infty$  may be viewed (compare with (1.13)) as a fractional Carleson measure condition ( $L^p$ -based, of order  $\eta$ ). In particular, it implies that the measure  $d\mu(x', t) := |\nabla u(x', t)|^2 t dt dx'$  satisfies the vanishing condition (1.11), with a precisely quantified rate of decay.

Here is the statement of the theorem advertised earlier which deals with the larger, more inclusive context considered above and which complements the end-point case  $\eta = 0$  corresponding to the portion of Theorem 1.1 pertaining to the well-posedness of the BMO-Dirichlet boundary value problem.

**Theorem 1.9.** *Let  $L$  be an  $M \times M$  elliptic constant complex coefficient system as in (1.1)-(1.2), and fix  $\eta \in (0, 1)$  along with  $p, q \in [1, \infty)$ . Then the Morrey-Campanato-Dirichlet boundary value problem for  $L$  in  $\mathbb{R}_+^n$ , formulated as*

$$\begin{cases} u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M), \\ Lu = 0 \text{ in } \mathbb{R}_+^n, \\ \|u\|_{**}^{(\eta, q)} < \infty, \\ u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} = f \text{ a.e. in } \mathbb{R}^{n-1}, f \in \mathcal{E}^{\eta, p}(\mathbb{R}^{n-1}, \mathbb{C}^M), \end{cases} \quad (1.70)$$

has a unique solution. The solution  $u$  of (1.70) is given by (1.25) and there exists a constant  $C = C(n, L, \eta, p, q) \in (1, \infty)$  with the property that

$$C^{-1} \|f\|_*^{(\eta, p)} \leq \|u\|_{**}^{(\eta, q)} \leq C \|f\|_*^{(\eta, p)}. \quad (1.71)$$

Moreover,  $u$  belongs to  $\dot{\mathcal{C}}^\eta(\mathbb{R}_+^n, \mathbb{C}^M) = \dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)$  and, with  $C \in (1, \infty)$  as above,

$$C^{-1} \|f\|_*^{(\eta, p)} \leq \|u\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}_+^n, \mathbb{C}^M)} \leq C \|f\|_*^{(\eta, p)}. \quad (1.72)$$

As a consequence of Theorem 1.9 and its proof (cf. also (2.2)) we obtain that, in fact,

$$\mathcal{E}^{\eta, p}(\mathbb{R}^{n-1}, \mathbb{C}^M) = \dot{\mathcal{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M) \quad (1.73)$$

as vector spaces, with equivalent norms. This offers a new proof (of a PDE flavor) of an old embedding result of N. Meyers [25]. An inspection of the proof of Theorem 1.9 also reveals that there is a Fatou type result naturally accompanying the well-posedness result for the boundary value problem (1.70).

We shall now succinctly comment on the literature dealing with Dirichlet boundary value problems for elliptic operators in the upper-half space. As already noted, the nature of these problems strongly depends on the choice of the function space from which the boundary datum  $f$  is selected, the specific way in which the size of the solution  $u$  is measured, and the very manner in which its boundary trace is considered. To illustrate these distinctions, recall first that there is a vast body of work targeting the case when the solution  $u$  is sought in various Sobolev spaces in  $\mathbb{R}_+^n$ , the boundary datum  $f$  belongs to suitable Besov spaces on  $\mathbb{R}^{n-1}$ , and the boundary trace of  $u$  is considered in the sense of Sobolev space theory. Classical references in this regard include [1], [2], [17], [22], [37], and the reader is also referred to the literature cited therein.

The scenario in which the size of  $u$  is measured in terms of the nontangential maximal operator (1.64) and when the trace of  $u$  on the boundary of  $\mathbb{R}_+^n$  is taken in a nontangential pointwise sense (cf. (1.9)) has been treated in [18] for the general class of  $M \times M$  systems  $L$  with constant complex coefficients as in (1.1)-(1.2). This extends classical work carried out in the particular case when  $L = \Delta$ , the Laplacian in  $\mathbb{R}^n$ , treated in a number of monographs, including [4], [14], [34], [35], and [36]. The corresponding higher-order regularity Dirichlet problem in a similar framework has been recently considered in [19]. See also [20] for related work, emphasizing semigroup techniques.

There is also a significant amount of work focused on the classical Dirichlet problem for the Laplacian in the upper-half space with a continuous boundary datum  $f$ . In such a case, one seeks a harmonic function  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n) \cap \mathcal{C}^0(\overline{\mathbb{R}_+^n})$  satisfying  $u|_{\partial\mathbb{R}_+^n} = f$ . A remarkable feature (noted by Helms in [15, p. 42 and p. 158]) is that even in the case when the boundary datum  $f$  is a bounded continuous function in  $\mathbb{R}^{n-1}$  the solution  $u$  of this classical Dirichlet problem is not unique. To ensure uniqueness in such a setting one typically specifies the behavior of  $u(x', t)$  as  $t \rightarrow \infty$ . A case in point is [30], where uniqueness is established in the class of harmonic functions  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n) \cap \mathcal{C}^0(\overline{\mathbb{R}_+^n})$  satisfying  $u(x) = o(|x| \sec^\gamma \theta)$  as  $|x| \rightarrow \infty$  (where  $\theta := \arccos(x_n/|x|)$  and  $\gamma \in \mathbb{R}$  is arbitrary), by proving a Phragmen-Lindelöf principle under the latter growth condition. This builds on the work of [29], [39], [38], and others. The works just cited crucially rely on positivity and other various highly specialized properties of the Laplace operator, so the techniques employed there do not adapt to the considerably more general class of elliptic systems considered in the present paper.

In relation to the context just described above it is instructive to make the following observations. First, the collection of uniformly continuous functions belonging to  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  is a dense subspace of  $\text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  (see (1.22)). Second, in the last part of Theorem 1.1 we have succeeded in proving the well-posedness of the VMO-Dirichlet problem in the class of

null-solutions  $u$  of a given elliptic system  $L$  as in (1.1)-(1.2) which satisfy a vanishing Carleson measure condition. This is a natural condition from the point of view of harmonic analysis which replaces the demand that the solution extends continuously on  $\overline{\mathbb{R}_+^n}$ , required in the formulation of the classical Dirichlet problem with continuous data.

Apparently, the closest results in the literature to some of the work carried out in this paper are those of E. Fabes, R. Johnson, and U. Neri from 1976. Indeed, in [11] these authors have dealt with the BMO-Dirichlet problem for the Laplacian in the upper-half space in the class of harmonic functions satisfying a Carleson measure condition (this being said, we would like to point out that there are certain gaps in some of the key steps of the treatment in [11], such as the proof of Lemma 1.3 on pp. 161–162, or the proof of the estimate (1.5) on page 163). The portion of Theorem 1.1 dealing with (1.24) is a significant generalization of their work, which is thereby extended to a much larger class of systems. Similar attributes are shared by our Theorem 1.9 in relation to the work in [11] dealing with harmonic functions in the upper-half space with traces in Morrey-Campanato spaces.

We also wish to mention here the work of B. Dahlberg and C. Kenig who have treated the BMO-Dirichlet problem for the Laplacian in [9, Theorem 4.18, p. 463] in bounded Lipschitz domains via layer potentials, building on the earlier work of E. Fabes and U. Neri from [12] who employed harmonic measure techniques. For related work see also [10].

The techniques employed in [9], [10], [11], [12], are largely restricted to scalar equations (as they make essential use of positivity and/or maximum principles). Also, the fact that in [9], [10], [12], the underlying domain is bounded makes the task of proving uniqueness considerably more manageable. In addition, the consideration of PDE's for which the well-posedness of the  $L^2$ -Dirichlet problem is known in arbitrary Lipschitz subdomains allows these authors to successfully employ a variety of localizations arguments. By way of contrast, most of these key features cease to be effective in the geometric/analytic context considered in this paper. In proving the solvability of the BMO-Dirichlet boundary value problem for an elliptic system  $L$  in  $\mathbb{R}_+^n$  as formulated in (1.24), our approach makes essential use of the existence and properties of the Poisson kernel associated with  $L$  from the work of [1]-[2]. Uniqueness is derived with the help of the Fatou type result recorded in Theorem 1.2. A considerable amount of effort then goes into establishing the latter theorem, with square-function estimates (cf. Proposition 3.2), elements of tent-space theory (Lemma 4.10), interior estimates (cf. Theorem 2.4), and certain estimates near the boundary from [23] for null-solutions of  $L$  vanishing on the boundary (cf. Proposition 2.5), among the tools playing a key role in this regard.

We conclude with a brief overview of the contents of the sections of this paper. Useful background material and auxiliary results are collected in §2. The proofs of the existence statements in Theorem 1.1, both for the BMO-Dirichlet problem and the VMO-Dirichlet problem, are carried out in §3. Next, §4 is reserved for establishing a Fatou result for smooth null-solutions of  $L$  satisfying a Carleson measure condition, as well as uniqueness in the BMO-Dirichlet problem, in the upper-half space. Finally, the proofs of Theorems 1.1-1.4, Theorem 1.6, and Theorem 1.8 are given in §5, while the proof of Theorem 1.9 is contained in §6.

## 2. BACKGROUND MATERIAL AND PRELIMINARY RESULTS

In this section we collect a number of preliminary results that are useful in the sequel. Throughout, we let  $\mathbb{N}$  stand for the collection of all positive integers, and set  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ . In this way  $\mathbb{N}_0^k$  stands for the set of multi-indices  $\alpha = (\alpha_1, \dots, \alpha_k)$  with  $\alpha_j \in \mathbb{N}_0$  for  $1 \leq j \leq k$ . Also, fix  $n \in \mathbb{N}$  with  $n \geq 2$ . For an arbitrary multi-index  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$  we use the standard notation  $\partial^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$  and we occasionally abbreviate  $\partial_{x_j}$  by simply  $\partial_j$  for  $j \in \{1, \dots, n\}$ . The length of the multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$  is defined as  $|\alpha| := \alpha_1 + \dots + \alpha_n$ . We agree to let  $\{e_j\}_{1 \leq j \leq n}$  stand for the standard orthonormal basis in  $\mathbb{R}^n$ . Occasionally, we

canonically identify  $e_j$  with a multi-index in  $\mathbb{N}_0$  (of length 1). Given an arbitrary set  $E \subseteq \mathbb{R}^{n-1}$  we denote by  $\mathbf{1}_E$  the characteristic function of  $E$ .

Generally speaking, given a metric space  $(X, d)$ , corresponding to each subset  $E$  of  $X$  (of cardinality  $\geq 2$ ) and number  $\eta > 0$ , we associate the homogeneous Hölder space or order  $\eta$ , denoted by  $\dot{\mathcal{C}}^\eta(E, \mathbb{C}^M)$ , as the collection of functions  $w : E \rightarrow \mathbb{C}^M$  satisfying

$$\|w\|_{\dot{\mathcal{C}}^\eta(E, \mathbb{C}^M)} := \sup_{\substack{x, y \in E \\ x \neq y}} \frac{|w(x) - w(y)|}{d(x, y)^\eta} < \infty. \quad (2.1)$$

Whenever  $E \subseteq F \subseteq X$  (with  $E$  having cardinality  $\geq 2$ ) we then have

$$\begin{aligned} \dot{\mathcal{C}}^\eta(E, \mathbb{C}^M) &= \dot{\mathcal{C}}^\eta(\overline{E}, \mathbb{C}^M) \text{ isometrically, and} \\ \dot{\mathcal{C}}^\eta(F, \mathbb{C}^M) \ni w &\mapsto w|_E \in \dot{\mathcal{C}}^\eta(E, \mathbb{C}^M) \text{ continuously.} \end{aligned} \quad (2.2)$$

Also,

$$\dot{\mathcal{C}}^\eta(E, \mathbb{C}^M) \subseteq \text{UC}(E, \mathbb{C}^M), \quad (2.3)$$

where the latter denotes the space of  $\mathbb{C}^M$ -valued functions which are uniformly continuous on the set  $E$ . Finally, we agree to drop the dependence on the range when  $M = 1$ , and denote by  $\text{Lip}(E)$  the homogeneous Hölder space on  $E$  of order  $\eta = 1$ .

Moving on, we denote by  $\mathcal{M}$  the Hardy-Littlewood maximal operator on  $\mathbb{R}^{n-1}$  which acts on vector-valued functions with components in  $L^1_{\text{loc}}(\mathbb{R}^{n-1})$  according to

$$(\mathcal{M}f)(x') := \sup_{Q \ni x'} \int_Q |f(y')| dy', \quad \forall x' \in \mathbb{R}^{n-1}, \quad (2.4)$$

where the supremum runs over all cubes  $Q$  in  $\mathbb{R}^{n-1}$  containing  $x'$ .

We will often work with the weighted Lebesgue space of the form

$$\begin{aligned} L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1 + |x'|^a}\right) \\ := \left\{ f : \mathbb{R}^{n-1} \rightarrow \mathbb{C} \text{ Lebesgue measurable} : \int_{\mathbb{R}^{n-1}} \frac{|f(x')|}{1 + |x'|^a} dx' < \infty \right\}, \end{aligned} \quad (2.5)$$

where  $a \in (0, \infty)$ , and we shall denote by  $L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1 + |x'|^a}\right)^M$  the space of  $\mathbb{C}^M$ -valued functions with components in (2.5). Clearly,

$$L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1 + |x'|^a}\right)^M \subset L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M), \quad \forall a > 0. \quad (2.6)$$

Next, we record several useful properties of mean oscillations (recall the piece of notation introduced in (1.15)). First we note that if  $Q$  and  $Q'$  are cubes in  $\mathbb{R}^{n-1}$  with the property that  $Q' \subseteq Q$ , then for any  $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  and any  $p \in [1, \infty)$  we have

$$\left( \int_{Q'} |f(y') - f_{Q'}|^p dy' \right)^{\frac{1}{p}} \leq 2 \left( \frac{\ell(Q)}{\ell(Q')} \right)^{\frac{n-1}{p}} \left( \int_Q |f(y') - f_Q|^p dy' \right)^{\frac{1}{p}} \quad (2.7)$$

and

$$\left( \int_Q |f(y') - f_Q|^p dy' \right)^{\frac{1}{p}} \leq \left[ 1 + \left( \frac{\ell(Q)}{\ell(Q')} \right)^{\frac{n-1}{p}} \right] \left( \int_{Q'} |f(y') - f_{Q'}|^p dy' \right)^{\frac{1}{p}}. \quad (2.8)$$

Also,

$$\frac{1}{2} \left( \int_Q |f(y') - f_Q|^p dy' \right)^{\frac{1}{p}} \leq \inf_{c \in \mathbb{C}^M} \left( \int_Q |f(y') - c|^p dy' \right)^{\frac{1}{p}} \leq \left( \int_Q |f(y') - f_Q|^p dy' \right)^{\frac{1}{p}}. \quad (2.9)$$



Second, we recall the John-Nirenberg inequality asserting that there exist two dimensional constants  $C_1, C_2 \in (0, \infty)$  with the following significance. Consider an arbitrary cube  $Q \subset \mathbb{R}^{n-1}$  along with a function  $f \in L^1(Q)$  with the property that

$$N_Q(f) := \sup_{Q' \subseteq Q} \int_{Q'} |f(y') - f_{Q'}| dy' < \infty, \quad (2.10)$$

where the above supremum involves cubes  $Q' \subset \mathbb{R}^{n-1}$  contained in  $Q$ . Then there holds (see, e.g., [35, Corollary 2, p. 154])

$$\mathcal{L}^{n-1}(\{y' \in Q : |f(y') - f_Q| > \lambda\}) \leq C_1 e^{-\left(\frac{C_2}{N_Q(f)}\right)\lambda} |Q|, \quad \forall \lambda > 0. \quad (2.11)$$

Third, as a corollary of the John-Nirenberg inequality, we obtain that for every  $p \in (0, \infty)$  there exists a constant  $C_{n,p} \in (0, \infty)$  with the property that for every cube  $Q \subset \mathbb{R}^{n-1}$  and every function  $f \in L^1(Q, \mathbb{C}^M)$  we have

$$\left( \int_Q |f(y') - f_Q|^p dy' \right)^{\frac{1}{p}} \leq C_{n,p} \sup_{Q' \subseteq Q} \int_{Q'} |f(y') - f_{Q'}| dy'. \quad (2.12)$$

To proceed, for each  $p \in [1, \infty)$ ,  $r \in (0, \infty)$ , and  $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  define the  $L^p$ -based mean oscillations of  $f$  at a given scale  $r \in (0, \infty)$  as

$$\text{osc}_p(f; r) := \sup_{Q \subset \mathbb{R}^{n-1}, \ell(Q) \leq r} \left( \int_Q |f(x') - f_Q|^p dx' \right)^{\frac{1}{p}} \in [0, \infty]. \quad (2.13)$$

Some of the main properties of this function are summarized next.

**Lemma 2.1.** *For each  $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  the following properties hold.*

- (a) *Fix  $p \in [1, \infty)$ . Then, as a function of  $r$ , the quantity  $\text{osc}_p(f; r)$  is nondecreasing in  $r$ .*
- (b) *For every  $p, q \in [1, \infty)$  there exists a constant  $C = C(p, q, n) \in (1, \infty)$ , independent of  $f$ , with the property that*

$$C^{-1} \text{osc}_p(f; r) \leq \text{osc}_q(f; r) \leq C \text{osc}_p(f; r) \quad \text{for every } r \in (0, \infty). \quad (2.14)$$

- (c) *The function  $f$  actually belongs to  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  if and only if  $\text{osc}_p(f; r)$  as a function in  $r$  is bounded on  $(0, \infty)$  for some, or any,  $p \in [1, \infty)$ . Moreover, for each  $p \in [1, \infty)$  there exists a constant  $C = C(n, p) \in (1, \infty)$ , independent of  $f$ , with the property that*

$$C^{-1} \|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq \sup_{r>0} \text{osc}_p(f; r) = \lim_{r \rightarrow \infty} \text{osc}_p(f; r) \leq C \|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)}. \quad (2.15)$$

- (d) *The function  $f$  actually belongs to  $\text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  if and only if for some, or any  $p \in [1, \infty)$  one has*

$$\lim_{r \rightarrow 0^+} \text{osc}_p(f; r) = 0 \quad \text{and} \quad \lim_{r \rightarrow \infty} \text{osc}_p(f; r) < \infty. \quad (2.16)$$

- (e) *For every  $\eta \in [0, 1)$  and  $p \in [1, \infty)$  we have (recall (1.66))*

$$\text{osc}_p(f; r) \leq r^\eta \|f\|_*^{(\eta, p)}, \quad \forall r \in (0, \infty). \quad (2.17)$$

- (f) *If actually  $f$  belongs to  $\mathcal{C}^\Upsilon(\mathbb{R}^{n-1}, \mathbb{C}^M)$  for some modulus of continuity  $\Upsilon$  (recall (1.41)-(1.42)), then for each  $p \in [1, \infty)$  one has*

$$\text{osc}_p(f; r) \leq \|f\|_{\mathcal{C}^\Upsilon(\mathbb{R}^{n-1}, \mathbb{C}^M)} \Upsilon(\sqrt{n}r), \quad \forall r \in (0, \infty). \quad (2.18)$$

*In particular, for each  $p \in [1, \infty)$  and  $\eta \in (0, 1)$  there exists  $C \in (0, \infty)$  such that for every function  $f \in \mathcal{C}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)$  one has*

$$\text{osc}_p(f; r) \leq Cr^\eta \|f\|_{\mathcal{C}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)}, \quad \forall r \in (0, \infty). \quad (2.19)$$

*Proof.* The claim made in part (a) follows directly from (2.13). The claim in part (b) is a direct consequence of Hölder's inequality and John-Nirenberg's inequality (see (2.12)). The latter also implies the claims made in part (c). The claim in part (d) is a consequence of (a)-(c) and (1.21). Estimate (2.17) is immediate from (2.13) and (1.66). Finally, if actually  $f \in \mathcal{C}^\Upsilon(\mathbb{R}^{n-1}, \mathbb{C}^M)$  then for each  $p \in [1, \infty)$  and each cube  $Q$  in  $\mathbb{R}^{n-1}$  Hölder's inequality gives

$$\begin{aligned} \left( \int_Q |f(x') - f_Q|^p dx' \right)^{\frac{1}{p}} &\leq \left( \int_Q \int_Q |f(x') - f(y')|^p dy' dx' \right)^{\frac{1}{p}} \\ &\leq \|f\|_{\mathcal{C}^\Upsilon(\mathbb{R}^{n-1}, \mathbb{C}^M)} \Upsilon(\sqrt{n} \ell(Q)). \end{aligned} \quad (2.20)$$

Then (2.18) follows from (2.20) given that  $\Upsilon$  is nondecreasing.  $\square$

Next, we discuss the manner in which global integrability properties of a given function are related to the behavior at infinity of its mean oscillation function.

**Lemma 2.2.** *Fix  $\varepsilon > 0$  arbitrary. Then there exists a constant  $C_{n,\varepsilon} \in (0, \infty)$  such that for each function  $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  and each cube  $Q \subset \mathbb{R}^{n-1}$ , with center  $x'_Q \in \mathbb{R}^{n-1}$ , there holds*

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \frac{|f(y') - f_Q|}{[\ell(Q) + |x'_Q - y'|]^{n-1+\varepsilon}} dy' &\leq \frac{C_{n,\varepsilon}}{\ell(Q)^\varepsilon} \int_1^\infty \left( \int_{\lambda Q} |f(y') - f_{\lambda Q}| dy' \right) \frac{d\lambda}{\lambda^{1+\varepsilon}} \\ &\leq \frac{C_{n,\varepsilon}}{\ell(Q)^\varepsilon} \int_1^\infty \text{osc}_1(f; \lambda \ell(Q)) \frac{d\lambda}{\lambda^{1+\varepsilon}}. \end{aligned} \quad (2.21)$$

As a consequence, for each  $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  one has

$$\int_1^\infty \text{osc}_1(f; \lambda) \frac{d\lambda}{\lambda^{1+\varepsilon}} < \infty \implies f \in L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1 + |x'|^{n-1+\varepsilon}}\right)^M \quad (2.22)$$

and there exists a constant  $C_{n,\varepsilon} \in (0, \infty)$  with the property that

$$\int_{\mathbb{R}^{n-1}} \frac{|f(x')|}{1 + |x'|^{n-1+\varepsilon}} dx' \leq C_{n,\varepsilon} \int_1^\infty \text{osc}_1(f; \lambda) \frac{d\lambda}{\lambda^{1+\varepsilon}} + C_{n,\varepsilon} \int_{Q_0} |f(x')| dx' \quad (2.23)$$

where  $Q_0 := (-1/2, 1/2)^{n-1}$  is the cube centered at the origin  $0'$  of  $\mathbb{R}^{n-1}$  with side-length 1.

In particular, we have

$$\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) \subset L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1 + |x'|^{n-1+\varepsilon}}\right)^M, \quad \forall \varepsilon > 0, \quad (2.24)$$

and (recall (1.67))

$$\mathcal{E}^{\eta,p}(\mathbb{R}^{n-1}, \mathbb{C}^M) \subset L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1 + |x'|^{n-1+\varepsilon}}\right)^M, \quad \forall \varepsilon > 0, \quad \forall \eta \in [0, \varepsilon), \quad (2.25)$$

while in view of (2.19) and (2.22) we obtain

$$\mathcal{E}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M) \subset L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1 + |x'|^{n-1+\varepsilon}}\right)^M, \quad \forall \eta \in (0, \varepsilon). \quad (2.26)$$

*Proof.* Given  $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  and a cube  $Q \subset \mathbb{R}^{n-1}$  with center  $x'_Q \in \mathbb{R}^{n-1}$ , breaking up the domain of integration allows us to estimate

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \frac{|f(y') - f_Q|}{[\ell(Q) + |x'_Q - y'|]^{n-1+\varepsilon}} dy' \\ \leq \ell(Q)^{-n+1-\varepsilon} \int_Q |f(y') - f_Q| dy' + \sum_{k=0}^\infty \int_{2^{k+1}Q \setminus 2^k Q} \frac{|f(y') - f_Q|}{|x'_Q - y'|^{n-1+\varepsilon}} dy' \end{aligned}$$

$$\begin{aligned}
&\leq \ell(Q)^{-\varepsilon} \int_Q |f(y') - f_Q| dy' \\
&\quad + 2^{2(n-1)+\varepsilon} \ell(Q)^{-\varepsilon} \sum_{k=0}^{\infty} 2^{-k\varepsilon} \int_{2^{k+1}Q} |f(y') - f_Q| dy'. \tag{2.27}
\end{aligned}$$

Next, for each  $k \in \mathbb{N}_0$  we have

$$\begin{aligned}
\int_{2^{k+1}Q} |f(y') - f_Q| dy' &\leq \int_{2^{k+1}Q} |f(y') - f_{2^{k+1}Q}| dy' + \sum_{j=0}^k |f_{2^jQ} - f_{2^{j+1}Q}| \\
&\leq \int_{2^{k+1}Q} |f(y') - f_{2^{k+1}Q}| dy' \\
&\quad + 2^{n-1} \sum_{j=0}^k \int_{2^{j+1}Q} |f(y') - f_{2^{j+1}Q}| dy', \tag{2.28}
\end{aligned}$$

hence,

$$\begin{aligned}
\sum_{k=0}^{\infty} 2^{-k\varepsilon} \int_{2^{k+1}Q} |f(y') - f_Q| dy' &\leq \sum_{k=0}^{\infty} 2^{-k\varepsilon} \int_{2^{k+1}Q} |f(y') - f_{2^{k+1}Q}| dy' \\
&\quad + 2^{n-1} \sum_{k=0}^{\infty} 2^{-k\varepsilon} \left\{ \sum_{j=0}^k \int_{2^{j+1}Q} |f(y') - f_{2^{j+1}Q}| dy' \right\} \\
&= \sum_{k=0}^{\infty} 2^{-k\varepsilon} \int_{2^{k+1}Q} |f(y') - f_{2^{k+1}Q}| dy' \\
&\quad + \frac{2^{n-1}}{1-2^{-\varepsilon}} \sum_{j=0}^{\infty} 2^{-j\varepsilon} \int_{2^{j+1}Q} |f(y') - f_{2^{j+1}Q}| dy' \\
&= \left(1 + \frac{2^{n-1}}{1-2^{-\varepsilon}}\right) \sum_{k=0}^{\infty} 2^{-k\varepsilon} \int_{2^{k+1}Q} |f(y') - f_{2^{k+1}Q}| dy', \tag{2.29}
\end{aligned}$$

where the first equality has been obtained by interchanging the sums in  $k$  and  $j$ . Collectively, (2.27) and (2.29) permit us to conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} \frac{|f(y') - f_Q|}{[\ell(Q) + |x'_Q - y'|]^{n-1+\varepsilon}} dy' \\
&\leq 4^{n-1+\varepsilon} \left(1 + \frac{2^{n-1}}{1-2^{-\varepsilon}}\right) \ell(Q)^{-\varepsilon} \sum_{k=0}^{\infty} 2^{-k\varepsilon} \int_{2^kQ} |f(y') - f_{2^kQ}| dy'. \tag{2.30}
\end{aligned}$$

To proceed, observe that (2.7) yields

$$\begin{aligned}
\int_{2^kQ} |f(y') - f_{2^kQ}| dy' &\leq 2^n \int_{\lambda Q} |f(y') - f_{\lambda Q}| dy', \\
&\text{for each } k \in \mathbb{N}_0 \text{ and each } \lambda \in [2^k, 2^{k+1}]. \tag{2.31}
\end{aligned}$$

This, in turn, implies that for each  $k \in \mathbb{N}_0$  we have

$$2^{-k\varepsilon} \int_{2^kQ} |f(y') - f_{2^kQ}| dy' \leq \frac{2^n \varepsilon}{1-2^{-\varepsilon}} \int_{2^k}^{2^{k+1}} \left( \int_{\lambda Q} |f(y') - f_{\lambda Q}| dy' \right) \frac{d\lambda}{\lambda^{1+\varepsilon}}. \tag{2.32}$$

Availing ourselves of this estimate back into (2.30) then establishes the first inequality in (2.21) for the choice

$$C_{n,\varepsilon} := 2^n 4^{n-1+\varepsilon} \left(1 + \frac{2^{n-1}}{1-2^{-\varepsilon}}\right) \cdot \frac{\varepsilon}{1-2^{-\varepsilon}}. \quad (2.33)$$

The second inequality in (2.21) is a direct consequence of (2.13). Going further, (2.22)-(2.23) follow from the second inequality in (2.21) with  $Q := (-1/2, 1/2)^{n-1}$ . In turn, (2.23) together with part (c) in Lemma 2.1 give (2.24), while (2.23) together with part (e) in Lemma 2.1 give (2.25).  $\square$

Poisson kernels for elliptic operators in a half-space have a long history (see, e.g., [1], [2], [32], [33]). Here we record the following useful existence and uniqueness result. In its statement (as well as elsewhere in the paper), we make the convention that the convolution between two functions, which are matrix-valued and vector-valued, respectively, takes into account the algebraic multiplication between a matrix and a vector in a natural fashion.

**Theorem 2.3.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2). Then there exists a matrix-valued function  $P^L = (P_{\alpha\beta}^L)_{1 \leq \alpha, \beta \leq M} : \mathbb{R}^{n-1} \rightarrow \mathbb{C}^{M \times M}$  (called the Poisson kernel for  $L$  in  $\mathbb{R}_+^n$ ) satisfying the following properties:*

(1) *There exists  $C \in (0, \infty)$  such that*

$$|P^L(x')| \leq \frac{C}{(1 + |x'|^2)^{\frac{n}{2}}} \quad \text{for each } x' \in \mathbb{R}^{n-1}. \quad (2.34)$$

(2) *The function  $P^L$  is Lebesgue measurable and*

$$\int_{\mathbb{R}^{n-1}} P^L(x') dx' = I_{M \times M}, \quad (2.35)$$

where  $I_{M \times M}$  denotes the  $M \times M$  identity matrix. In particular, for every constant vector  $C = (C_\alpha)_{1 \leq \alpha \leq M} \in \mathbb{C}^M$  one has

$$\int_{\mathbb{R}^{n-1}} \sum_{1 \leq \beta \leq M} (P_{\alpha\beta}^L)_t(x' - y') C_\beta dy' = C_\alpha, \quad \forall (x', t) \in \mathbb{R}_+^n. \quad (2.36)$$

(3) *If one sets*

$$K^L(x', t) := P_t^L(x') = t^{1-n} P^L(x'/t) \quad \text{for each } x' \in \mathbb{R}^{n-1} \text{ and } t > 0, \quad (2.37)$$

*then the function  $K^L = (K_{\alpha\beta}^L)_{1 \leq \alpha, \beta \leq M}$  satisfies (in the sense of distributions)*

$$LK_{\cdot\beta}^L = 0 \quad \text{in } \mathbb{R}_+^n \text{ for each } \beta \in \{1, \dots, M\}, \quad (2.38)$$

where  $K_{\cdot\beta}^L := (K_{\alpha\beta}^L)_{1 \leq \alpha \leq M}$  is the  $\beta$ -th column in  $K^L$ .

Moreover,  $P^L$  is unique in the class of  $\mathbb{C}^{M \times M}$ -valued functions defined in  $\mathbb{R}^{n-1}$  and satisfying (1)-(3) above, and has the following additional properties:

(4) *One has  $P^L \in \mathcal{C}^\infty(\mathbb{R}^{n-1})$  and  $K^L \in \mathcal{C}^\infty(\overline{\mathbb{R}_+^n} \setminus B(0, \varepsilon))$  for every  $\varepsilon > 0$ . Consequently, (2.38) holds in a pointwise sense.*

(5) *There holds  $K^L(\lambda x) = \lambda^{1-n} K^L(x)$  for all  $x \in \mathbb{R}_+^n$  and  $\lambda > 0$ . In particular, for each multi-index  $\alpha \in \mathbb{N}_0^n$  there exists  $C_\alpha \in (0, \infty)$  with the property that*

$$|(\partial^\alpha K^L)(x)| \leq C_\alpha |x|^{1-n-|\alpha|}, \quad \forall x \in \overline{\mathbb{R}_+^n} \setminus \{0\}. \quad (2.39)$$

(6) For each  $\kappa > 0$  there exists a finite constant  $C_\kappa > 0$  with the property that for each  $x' \in \mathbb{R}^{n-1}$ ,

$$\sup_{|x'-y'| < \kappa t} |(P_t^L * f)(y')| \leq C_\kappa \mathcal{M}f(x'), \quad \forall f \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n} dx'\right)^M. \quad (2.40)$$

(7) Fix an arbitrary  $\kappa > 0$  and a function

$$f = (f_\beta)_{1 \leq \beta \leq M} \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n} dx'\right)^M. \quad (2.41)$$

Then the function  $u(x', t) := (P_t^L * f)(x')$  for each  $(x', t) \in \mathbb{R}_+^n$ , is meaningfully defined, via an absolutely convergent integral, satisfies

$$u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M), \quad Lu = 0 \quad \text{in } \mathbb{R}_+^n, \quad (2.42)$$

and, at every Lebesgue point  $x'_0 \in \mathbb{R}^{n-1}$  of  $f$ ,

$$\left(u \Big|_{\partial \mathbb{R}_+^n}^{\text{n.t.}}\right)(x'_0) := \lim_{\substack{(x', t) \rightarrow (x'_0, 0) \\ |x' - x'_0| < \kappa t}} (P_t^L * f)(x') = f(x'_0). \quad (2.43)$$

(8) The function  $P^L$  satisfies the semigroup property

$$P_{t_1}^L * P_{t_2}^L = P_{t_1+t_2}^L \quad \text{for every } t_1, t_2 > 0. \quad (2.44)$$

Concerning Theorem 2.3, we note that the existence part follows from the classical work of S. Agmon, A. Douglis, and L. Nirenberg in [2]. The uniqueness property has been recently proved in [18], where (2.40), (2.42), (2.43), as well as the semigroup property (2.44) have also been established.

Next, we record the following versatile version of interior estimates for higher-order elliptic systems. A proof may be found in [26, Theorem 11.9, p. 364].

**Theorem 2.4.** Assume the system  $L$  is as in (1.1)-(1.2). Then for each null-solution  $u$  of  $L$  in a ball  $B(x, R)$  (where  $x \in \mathbb{R}^n$  and  $R > 0$ ),  $p \in (0, \infty)$ ,  $\lambda \in (0, 1)$ ,  $\ell \in \mathbb{N}_0$ , and  $r \in (0, R)$ , one has

$$\sup_{z \in B(x, \lambda r)} |\nabla^\ell u(z)| \leq \frac{C}{r^\ell} \left( \int_{B(x, r)} |u|^p d\mathcal{L}^n \right)^{\frac{1}{p}}, \quad (2.45)$$

where  $C = C(L, p, \ell, \lambda, n) > 0$  is a finite constant.

To proceed we need to introduce some additional terminology. Let

$$W_{\text{bd}}^{1,2}(\mathbb{R}_+^n) := \{w \text{ Lebesgue measurable in } \mathbb{R}_+^n : \\ w, \nabla w \in L^2(\mathbb{R}_+^n \cap B(x, r)), \quad \forall x \in \mathbb{R}_+^n, \quad \forall r \in (0, \infty)\} \quad (2.46)$$

In the sequel, the space of  $\mathbb{C}^M$ -valued functions with components in  $W_{\text{bd}}^{1,2}(\mathbb{R}_+^n)$  will be denoted by  $W_{\text{bd}}^{1,2}(\mathbb{R}_+^n, \mathbb{C}^M)$ . Also, (whenever meaningful) the Sobolev trace  $\text{Tr}$  is defined as:

$$(\text{Tr } w)(x') := \lim_{r \rightarrow 0^+} \int_{B((x', 0), r) \cap \mathbb{R}_+^n} w d\mathcal{L}^n, \quad x' \in \mathbb{R}^{n-1}. \quad (2.47)$$

The following result can be found in [23, Corollary 2.4], and it is a consequence of the *a priori* regularity estimates obtained in [2] and Sobolev embeddings.

**Proposition 2.5.** *Let  $L$  be an  $M \times M$  elliptic system as in (1.1)-(1.2) and consider a vector-valued function  $w \in W_{\text{bd}}^{1,2}(\mathbb{R}_+^n, \mathbb{C}^M)$  such that*

$$\begin{cases} Lw = 0 & \text{in } \mathbb{R}_+^n, \\ \text{Tr } w = 0 & \mathcal{L}^{n-1}\text{-a.e. on } \mathbb{R}^{n-1}. \end{cases} \quad (2.48)$$

*Then  $w \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$ , and for each  $z \in \overline{\mathbb{R}_+^n}$  and  $\rho > 0$  one has*

$$\sup_{\mathbb{R}_+^n \cap B(z, \rho)} |\nabla w| \leq C \rho^{-1} \sup_{\mathbb{R}_+^n \cap B(z, 2\rho)} |w| \quad (2.49)$$

*where  $C \in (0, \infty)$  is a constant independent of the scale  $\rho$ , the point  $z$ , and the function  $w$ .*

We will also need an  $L^p$ -Fatou type result obtained in [18, Corollary 6.3]. To state it, the reader is invited to recall the nontangential maximal operator from (1.64).

**Corollary 2.6.** *Assume  $L$  is an elliptic  $M \times M$  system as in (1.1)-(1.2). Then for each  $p \in [1, \infty)$ ,*

$$\left. \begin{aligned} u &\in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M) \\ Lu &= 0 \text{ in } \mathbb{R}_+^n \\ \mathcal{N}u &\in L^p(\mathbb{R}^{n-1}) \end{aligned} \right\} \implies \begin{cases} u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} \text{ exists a.e. in } \mathbb{R}^{n-1}, \text{ belongs to } L^p(\mathbb{R}^{n-1}, \mathbb{C}^M), \\ \text{and } u(x', t) = \left( P_t^L * (u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}}) \right)(x'), \quad \forall (x', t) \in \mathbb{R}_+^n, \end{cases} \quad (2.50)$$

*where  $P^L$  is the Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.3.*

Our last auxiliary result, of a purely real-variable nature, can be found in [18, Lemma 3.3].

**Lemma 2.7.** *Fix  $M \in \mathbb{N}$  and let  $P = (P_{\alpha\beta})_{1 \leq \alpha, \beta \leq M} : \mathbb{R}^{n-1} \rightarrow \mathbb{C}^{M \times M}$  be a Lebesgue measurable function satisfying, for some  $c \in (0, \infty)$ ,*

$$|P(x')| \leq \frac{c}{(1 + |x'|^2)^{\frac{n}{2}}} \quad \text{for each } x' \in \mathbb{R}^{n-1}. \quad (2.51)$$

*Recall that  $P_t(x') = t^{1-n} P(x'/t)$  for each  $x' \in \mathbb{R}^{n-1}$  and  $t \in (0, \infty)$ .*

*Then, for each  $t \in (0, \infty)$  fixed, the operator*

$$L^1\left(\mathbb{R}^{n-1}, \frac{1}{1 + |x'|^n} dx'\right)^M \ni f \mapsto P_t * f \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1 + |x'|^n} dx'\right)^M \quad (2.52)$$

*is well-defined, linear and bounded, with operator norm controlled by  $C(t+1)$ . Moreover, for every  $\kappa > 0$  there exists a finite constant  $C_\kappa > 0$  with the property that for each  $x' \in \mathbb{R}^{n-1}$ ,*

$$\sup_{|x' - y'| < \kappa t} |(P_t * f)(y')| \leq C_\kappa \mathcal{M}f(x'), \quad \forall f \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1 + |x'|^n} dx'\right)^M. \quad (2.53)$$

*Finally, given any function*

$$f = (f_\beta)_{1 \leq \beta \leq M} \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1 + |x'|^n} dx'\right)^M \subset L_{\text{loc}}^1(\mathbb{R}^{n-1}, \mathbb{C}^M), \quad (2.54)$$

*at every Lebesgue point  $x'_0 \in \mathbb{R}^{n-1}$  of  $f$  there holds*

$$\lim_{\substack{(x', t) \rightarrow (x'_0, 0) \\ |x' - x'_0| < \kappa t}} (P_t * f)(x') = \left( \int_{\mathbb{R}^{n-1}} P(x') dx' \right) f(x'_0), \quad (2.55)$$

*and the function*

$$\mathbb{R}_+^n \ni (x', t) \mapsto (P_t * f)(x') \in \mathbb{C}^M \quad \text{is locally integrable in } \mathbb{R}_+^n. \quad (2.56)$$



## 3. PROOF OF THE EXISTENCE STATEMENTS IN THEOREM 1.1

This section is devoted to proving Proposition 3.1, dealing with the issue of existence for the BMO-Dirichlet boundary value problem (1.24), the upper estimate in (1.26), and the issue of existence for the VMO-Dirichlet boundary value problem (1.31).

In this regard, we find it useful to adopt a more general point of view, by going beyond the class BMO through the consideration of convolutions of the Poisson kernel with functions  $f$  from the weighted Lebesgue space  $L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n}\right)^M$  (recall the inclusion in (2.24)). The aforementioned convolutions are then shown to satisfy a variety of Carleson measure-like conditions, which only require (recall (2.13))

$$\int_1^\infty \text{osc}_1(f; \lambda) \frac{d\lambda}{\lambda^2} < \infty. \quad (3.1)$$

Note that this permits the oscillations  $\text{osc}_1(f; \lambda)$  of the given function  $f$  to grow with the scale  $\lambda$ . In particular, this allows us to simultaneously treat several scales of spaces of interest, including Hölder spaces  $\mathcal{C}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)$  with  $\eta \in (0, 1)$ , the Morrey-Campanato space  $\mathcal{E}^{\eta, p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  with  $\eta \in (0, 1)$  and  $p \in [1, \infty)$ , as well as the John-Nirenberg space  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ .

An example of a function  $f \in \mathcal{C}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)$  with  $\eta \in (0, 1)$  which does not belong to  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  is offered by

$$f(x') := |x'|^\eta, \quad \forall x' \in \mathbb{R}^{n-1}. \quad (3.2)$$

Indeed,  $\|f\|_{\mathcal{C}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq 1$  and since  $\delta_\lambda f = \lambda^\eta f$ , it follows from the last line in (1.16) that necessarily  $\|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} = \infty$ . Incidentally, for  $f$  as in (3.2), we have  $\text{osc}_1(f; \lambda) = O(\lambda^\eta)$  as  $\lambda \rightarrow \infty$ , hence (3.1) holds in this case.

Here is the formal statement of the result just advertised above.

**Proposition 3.1.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2) and let  $P^L$  be the Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.3. Select  $f \in L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n}\right)^M$  and set*

$$u(x', t) := (P_t^L * f)(x'), \quad \forall (x', t) \in \mathbb{R}_+^n. \quad (3.3)$$

*Then  $u$  is meaningfully defined via an absolutely convergent integral,*

$$u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M), \quad Lu = 0 \quad \text{in } \mathbb{R}_+^n, \quad \text{and } u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} = f \quad \text{a.e. in } \mathbb{R}^{n-1}. \quad (3.4)$$

*In addition,  $u$  enjoys the following properties:*

- (a) *For each integer  $\ell \geq 1$  there exists a constant  $C \in (0, \infty)$  with the property that the following pointwise estimate holds for every  $(x', t) \in \mathbb{R}_+^n$ :*

$$|(\nabla^\ell u)(x', t)| \leq \frac{C}{t^\ell} \int_1^\infty \text{osc}_1(f; \lambda t) \frac{d\lambda}{\lambda^{1+\ell}}. \quad (3.5)$$

*In particular, there exists  $C \in (0, \infty)$  such that*

$$|(\nabla u)(x', t)| \leq \frac{C}{t} \int_1^\infty \text{osc}_1(f; \lambda t) \frac{d\lambda}{\lambda^2}, \quad \forall (x', t) \in \mathbb{R}_+^n. \quad (3.6)$$

- (b) *There exists a constant  $C \in (0, \infty)$  such that for every cube  $Q$  in  $\mathbb{R}^{n-1}$  the following “cube-by-cube” Carleson measure estimates hold:*

$$\left( \int_0^{\ell(Q)} \int_Q |(\nabla u)(x', t)|^2 t \, dx' dt \right)^{\frac{1}{2}} \leq C \int_1^\infty \left( \int_{\lambda Q} |f(y') - f_{\lambda Q}| \, dy' \right) \frac{d\lambda}{\lambda^2}$$

$$+ C \sup_{Q' \subseteq 4Q} \int_{Q'} |f(y') - f_{Q'}| dy' \quad (3.7)$$

and

$$\left( \int_0^{\ell(Q)} \int_Q |(\nabla u)(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} \leq C \int_1^\infty \text{osc}_1(f; \lambda \ell(Q)) \frac{d\lambda}{\lambda^2}. \quad (3.8)$$

(c) *There exists  $C \in (0, \infty)$  such that the following local Carleson measure estimate holds for every scale  $r \in (0, \infty)$ :*

$$\sup_{Q \subset \mathbb{R}^{n-1}, \ell(Q) \leq r} \left( \int_0^{\ell(Q)} \int_Q |(\nabla u)(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} \leq C \int_1^\infty \text{osc}_1(f; r\lambda) \frac{d\lambda}{\lambda^2}. \quad (3.9)$$

(d) *Whenever  $f$  satisfies*

$$\int_1^\infty \text{osc}_1(f; \lambda) \frac{d\lambda}{\lambda^2} < \infty, \quad (3.10)$$

*the global weighted Carleson measure estimate*

$$\sup_{Q \subset \mathbb{R}^{n-1}} \left\{ \left( \int_1^\infty \text{osc}_1(f; \lambda \ell(Q)) \frac{d\lambda}{\lambda^2} \right)^{-1} \left( \int_0^{\ell(Q)} \int_Q |(\nabla u)(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} \right\} \leq C \quad (3.11)$$

*holds for some  $C \in (0, \infty)$  independent of  $f$ .*

(e) *There exists a constant  $C \in (0, \infty)$  such that the following global Carleson measure estimate holds:*

$$\|u\|_{**} = \sup_{Q \subset \mathbb{R}^{n-1}} \left( \int_0^{\ell(Q)} \int_Q |(\nabla u)(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} \leq C \|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)}. \quad (3.12)$$

*In particular, thanks to (2.24), estimate (3.12) holds for every  $f \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ .*

(f) *Whenever  $f$  satisfies*

$$\int_1^\infty \text{osc}_1(f; \lambda) \frac{d\lambda}{\lambda^2} < \infty \quad \text{and} \quad \lim_{r \rightarrow 0^+} \text{osc}_1(f; r) = 0, \quad (3.13)$$

*the following vanishing Carleson measure condition holds:*

$$\lim_{r \rightarrow 0^+} \left\{ \sup_{Q \subset \mathbb{R}^{n-1}, \ell(Q) \leq r} \left( \int_0^{\ell(Q)} \int_Q |(\nabla u)(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} \right\} = 0. \quad (3.14)$$

*In particular, in the case when actually  $f \in \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  to begin with,  $u$  has the additional property that*

$$|\nabla u(x', t)|^2 t dx' dt \text{ is a vanishing Carleson measure in } \mathbb{R}_+^n. \quad (3.15)$$

Ultimately, the proof of Proposition 3.1 relies on square-function estimates. For now, assuming a suitable  $L^2$  bound (implicit in (3.19) below) we may establish some versatile Carleson measure estimates (of local and global nature), as well as vanishing Carleson measure properties for integral operators (modeled upon the gradient of the convolution with the Poisson kernel) acting on function spaces larger than the standard BMO. This is made precise in the following proposition.

**Proposition 3.2.** *Let  $\theta : \mathbb{R}_+^n \times \mathbb{R}^{n-1} \longrightarrow \mathbb{C}^{M \times M}$  be a matrix-valued Lebesgue measurable function, with the property that there exist  $\varepsilon \in (0, \infty)$  and  $C \in (0, \infty)$  such that*

$$|\theta(x', t; y')| \leq \frac{C t^\varepsilon}{|(x' - y', t)|^{n-1+\varepsilon}}, \quad \forall (x', t) \in \mathbb{R}_+^n, \quad \forall y' \in \mathbb{R}^{n-1}, \quad (3.16)$$

and the following cancellation condition holds:

$$\int_{\mathbb{R}^{n-1}} \theta(x', t; y') dy' = 0 \in \mathbb{C}^{M \times M}, \quad \forall (x', t) \in \mathbb{R}_+^n. \quad (3.17)$$

In relation to the kernel  $\theta$ , one may then consider the integral operator  $\Theta$  acting on arbitrary functions  $f \in L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1+\varepsilon}}\right)^M$  according to (the absolutely convergent integral)

$$(\Theta f)(x', t) := \int_{\mathbb{R}^{n-1}} \theta(x', t; y') f(y') dy', \quad \forall (x', t) \in \mathbb{R}_+^n. \quad (3.18)$$

Then, under the assumption that the operator

$$\Theta : L^2(\mathbb{R}^{n-1}, \mathbb{C}^M) \longrightarrow L^2\left(\mathbb{R}_+^n, \frac{dx' dt}{t}\right)^M \text{ is bounded,} \quad (3.19)$$

the following properties hold:

- (a) *There exists a constant  $C \in (0, \infty)$  such that for every  $f \in L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1+\varepsilon}}\right)^M$  and every cube  $Q$  in  $\mathbb{R}^{n-1}$  the following “cube-by-cube” Carleson measure estimates hold:*

$$\begin{aligned} \left( \int_0^{\ell(Q)} \int_Q |(\Theta f)(x', t)|^2 \frac{dx' dt}{t} \right)^{\frac{1}{2}} &\leq C \int_1^\infty \left( \int_{\lambda Q} |f(y') - f_{\lambda Q}| dy' \right) \frac{d\lambda}{\lambda^{1+\varepsilon}} \\ &\quad + C \sup_{Q' \subseteq 4Q} \int_{Q'} |f(y') - f_{Q'}| dy' \end{aligned} \quad (3.20)$$

and

$$\left( \int_0^{\ell(Q)} \int_Q |(\Theta f)(x', t)|^2 \frac{dx' dt}{t} \right)^{\frac{1}{2}} \leq C \int_1^\infty \text{osc}_1(f; \lambda \ell(Q)) \frac{d\lambda}{\lambda^{1+\varepsilon}}. \quad (3.21)$$

- (b) *There exists  $C \in (0, \infty)$  such that for every function  $f \in L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1+\varepsilon}}\right)^M$  the following local Carleson measure estimate holds for every scale  $r \in (0, \infty)$ :*

$$\sup_{Q \subset \mathbb{R}^{n-1}, \ell(Q) \leq r} \left( \int_0^{\ell(Q)} \int_Q |(\Theta f)(x', t)|^2 \frac{dx' dt}{t} \right)^{\frac{1}{2}} \leq C \int_1^\infty \text{osc}_1(f; r\lambda) \frac{d\lambda}{\lambda^{1+\varepsilon}}. \quad (3.22)$$

- (c) *There exists  $C \in (0, \infty)$  such that for any given function  $f \in L_{\text{loc}}^1(\mathbb{R}^{n-1}, \mathbb{C}^M)$  with the property that*

$$\int_1^\infty \text{osc}_1(f; \lambda) \frac{d\lambda}{\lambda^{1+\varepsilon}} < \infty \quad (3.23)$$

(which necessarily places  $f$  into the space  $L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1+\varepsilon}}\right)^M$  by (2.23)) the following global weighted Carleson measure estimate holds:

$$\sup_{Q \subset \mathbb{R}^{n-1}} \left\{ \left( \int_1^\infty \text{osc}_1(f; \lambda \ell(Q)) \frac{d\lambda}{\lambda^{1+\varepsilon}} \right)^{-1} \left( \int_0^{\ell(Q)} \int_Q |(\Theta f)(x', t)|^2 \frac{dx' dt}{t} \right)^{\frac{1}{2}} \right\} \leq C. \quad (3.24)$$

(d) *There exists a constant  $C \in (0, \infty)$  such that for every  $f \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  the following global Carleson measure estimate holds:*

$$\sup_{Q \subset \mathbb{R}^{n-1}} \left( \int_0^{\ell(Q)} \int_Q |(\Theta f)(x', t)|^2 \frac{dx' dt}{t} \right)^{\frac{1}{2}} \leq C \|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)}. \quad (3.25)$$

(e) *Whenever  $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  is such that*

$$\int_1^\infty \text{osc}_1(f; \lambda) \frac{d\lambda}{\lambda^{1+\varepsilon}} < \infty \quad \text{and} \quad \lim_{r \rightarrow 0^+} \text{osc}_1(f; r) = 0, \quad (3.26)$$

*then  $f \in L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1+\varepsilon}}\right)^M$  and the following vanishing Carleson measure condition holds:*

$$\lim_{r \rightarrow 0^+} \left\{ \sup_{Q \subset \mathbb{R}^{n-1}, \ell(Q) \leq r} \left( \int_0^{\ell(Q)} \int_Q |(\Theta f)(x', t)|^2 \frac{dx' dt}{t} \right)^{\frac{1}{2}} \right\} = 0. \quad (3.27)$$

*In particular, (3.27) holds for every function  $f \in \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ .*

*Proof.* Start by fixing an arbitrary cube  $Q$  in  $\mathbb{R}^{n-1}$  and denote by  $x'_Q$  its center. Given a function  $f \in L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1+\varepsilon}}\right)^M$ , use (3.17) in order to write

$$\begin{aligned} \left( \int_0^{\ell(Q)} \int_Q |(\Theta f)(x', t)|^2 \frac{dx' dt}{t} \right)^{\frac{1}{2}} &= \left( \int_0^{\ell(Q)} \int_Q |(\Theta(f - f_Q))(x', t)|^2 \frac{dx' dt}{t} \right)^{\frac{1}{2}} \\ &\leq I + II, \end{aligned} \quad (3.28)$$

where

$$I := \left( \int_0^{\ell(Q)} \int_Q |\Theta((f - f_Q) \mathbf{1}_{4Q})(x', t)|^2 \frac{dx' dt}{t} \right)^{\frac{1}{2}} \quad (3.29)$$

and

$$II := \left( \int_0^{\ell(Q)} \int_Q |\Theta((f - f_Q) \mathbf{1}_{\mathbb{R}^{n-1} \setminus 4Q})(x', t)|^2 \frac{dx' dt}{t} \right)^{\frac{1}{2}}. \quad (3.30)$$

To estimate  $I$ , invoke (3.19), (2.8) with  $p = 2$ , and (2.12) to estimate

$$\begin{aligned} I &\leq \frac{1}{|Q|^{\frac{1}{2}}} \left( \int_{\mathbb{R}_+^n} |\Theta((f - f_Q) \mathbf{1}_{4Q})(x', t)|^2 \frac{dx' dt}{t} \right)^{\frac{1}{2}} \\ &\leq C \left( \int_{4Q} |f(y') - f_Q|^2 dy' \right)^{\frac{1}{2}} \leq C \left( \int_{4Q} |f(y') - f_{4Q}|^2 dy' \right)^{\frac{1}{2}} \\ &\leq C \sup_{Q' \subseteq 4Q} \int_{Q'} |f(y') - f_{Q'}| dy', \end{aligned} \quad (3.31)$$

where  $C \in (0, \infty)$  is independent of  $f$  and  $Q$ . To proceed, observe that there exists a purely dimensional constant  $c_n \in (0, \infty)$  (e.g., the choice  $c_n := 3/(6 + 2\sqrt{n-1})$  will do) with the property that

$$|x' - y'| \geq c_n(\ell(Q) + |x'_Q - y'|) \quad \text{for each } x' \in Q, y' \in \mathbb{R}^{n-1} \setminus 4Q. \quad (3.32)$$

Based on this and (3.18), (3.16), we may then estimate

$$|\Theta((f - f_Q) \mathbf{1}_{\mathbb{R}^{n-1} \setminus 4Q})(x', t)| \leq C t^\varepsilon \int_{\mathbb{R}^{n-1}} \frac{|f(y') - f_Q|}{[\ell(Q) + |x'_Q - y'|]^{n-1+\varepsilon}} dy', \quad (3.33)$$

for every point  $x' \in Q$  and every number  $t > 0$ ,

for some  $C \in (0, \infty)$  depending only on  $n$  and the constant appearing in (3.16). In turn, from (3.33) and (2.21) we conclude that

$$\begin{aligned} II &\leq C \left( \int_0^{\ell(Q)} \left( \frac{t}{\ell(Q)} \right)^{2\varepsilon} \frac{dt}{t} \right)^{\frac{1}{2}} \cdot \int_1^\infty \left( \int_{\lambda Q} |f(y') - f_{\lambda Q}| dy' \right) \frac{d\lambda}{\lambda^{1+\varepsilon}} \\ &= C \int_1^\infty \left( \int_{\lambda Q} |f(y') - f_{\lambda Q}| dy' \right) \frac{d\lambda}{\lambda^{1+\varepsilon}}. \end{aligned} \quad (3.34)$$

At this stage, (3.28), (3.31), and (3.34) combine to give (3.20). In turn, (3.21) readily follows from (3.20) and part (a) in Lemma 2.1 which allows us to estimate

$$\begin{aligned} \text{osc}_1(f; 4\ell(Q)) &\leq \varepsilon (4^{-\varepsilon} - 5^{-\varepsilon})^{-1} \int_4^5 \text{osc}_1(f; \lambda \ell(Q)) \frac{d\lambda}{\lambda^{1+\varepsilon}} \\ &\leq C_\varepsilon \int_1^\infty \text{osc}_1(f; \lambda \ell(Q)) \frac{d\lambda}{\lambda^{1+\varepsilon}}. \end{aligned} \quad (3.35)$$

In concert with part (a) in Lemma 2.1, estimate (3.21) immediately gives (3.22). Estimate (3.21) also implies the global weighted Carleson measure estimate formulated in (3.24). From (3.22) and part (c) in Lemma 2.1, the global Carleson measure estimate stated in (3.25) follows.

Going further, assume the function  $f \in L^1_{\text{loc}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  satisfies the properties listed in (3.26). Then  $f \in L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^{n-1+\varepsilon}}\right)^M$  by (2.22). Also, thanks to (3.26) and part (a) in Lemma 2.1, Lebesgue's Dominated Convergence Theorem applies and yields

$$\lim_{r \rightarrow 0^+} \int_1^\infty \text{osc}_1(f; r\lambda) \frac{d\lambda}{\lambda^{1+\varepsilon}} = 0. \quad (3.36)$$

Together with (3.22), this ultimately proves the vanishing Carleson measure condition stated in (3.27). Finally, that any function  $f \in \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  actually satisfies the properties listed in (3.26) is clear from (1.21), (2.13), and part (c) in Lemma 2.1. This completes the proof of Proposition 3.2.  $\square$

Next the goal is to identify a class of integral kernels  $\theta$  satisfying (3.16)-(3.17) with the property that the operator  $\Theta$  associated with  $\theta$  as in (3.18) enjoys the  $L^2$ -boundedness condition formulated in (3.19). We adopt a broader point of view by considering a larger variety of spaces, which turns out to be useful later. To set the stage, let us recall the definition of the Hardy space  $H^1(\mathbb{R}^{n-1})$  using  $(1, \infty)$ -atoms. Specifically, a Lebesgue measurable function  $a : \mathbb{R}^{n-1} \rightarrow \mathbb{C}$  is said to be a  $(1, \infty)$ -atom provided there exists a cube  $Q \subset \mathbb{R}^{n-1}$  such that the following localization, normalization, and cancellation properties hold:

$$\text{supp } a \subseteq Q, \quad \|a\|_{L^\infty(\mathbb{R}^{n-1})} \leq |Q|^{-1}, \quad \int_{\mathbb{R}^{n-1}} a(y') dy' = 0. \quad (3.37)$$

The space  $H^1(\mathbb{R}^{n-1})$  is then defined as the collection of all Lebesgue measurable functions  $f$  defined in  $\mathbb{R}^{n-1}$  such that

$$f = \sum_{j=1}^\infty \lambda_j a_j \quad \text{a.e. in } \mathbb{R}^{n-1}, \quad (3.38)$$

with the  $a_j$ 's being  $(1, \infty)$ -atoms, and where the sequence  $\{\lambda_j\}_{j \in \mathbb{N}} \subset \mathbb{C}$  satisfies  $\sum_{j=1}^{\infty} |\lambda_j| < \infty$ .

The norm in  $H^1(\mathbb{R}^{n-1})$  is defined as

$$\|f\|_{H^1(\mathbb{R}^{n-1})} := \inf \sum_{j=1}^{\infty} |\lambda_j|, \quad (3.39)$$

where the infimum runs over all the atomic decompositions of  $f$  as in (3.38). In particular, the series in (3.38) converges in  $H^1(\mathbb{R}^{n-1})$ . Let us also write  $H^1(\mathbb{R}^{n-1}, \mathbb{C}^M)$  for the collection of all  $\mathbb{C}^M$ -valued functions  $f = (f_\alpha)_{1 \leq \alpha \leq M}$  with components in  $H^1(\mathbb{R}^{n-1})$ . In such a scenario, we set

$$\|f\|_{H^1(\mathbb{R}^{n-1}, \mathbb{C}^M)} := \sum_{\alpha=1}^M \|f_\alpha\|_{H^1(\mathbb{R}^{n-1})}. \quad (3.40)$$

Here are the square-function estimates alluded to earlier. For more background and relevant references the reader is referred to the recent exposition in [28].

**Proposition 3.3.** *Let  $\theta$  and  $\Theta$  be as in (3.16)-(3.18) with  $\varepsilon = 1$  and  $M = 1$ . In addition, assume  $\theta$  is of class  $\mathcal{C}^1$  in the variable  $y' \in \mathbb{R}^{n-1}$  and suppose there exists some  $C \in (0, \infty)$  with the property that*

$$|\nabla_{y'} \theta(x', t; y')| \leq \frac{Ct}{|(x' - y', t)|^{n+1}}, \quad \forall (x', t) \in \mathbb{R}_+^n, \quad \forall y' \in \mathbb{R}^{n-1}. \quad (3.41)$$

Fix a background parameter  $\kappa > 0$  and, with the nontangential cone  $\Gamma_\kappa(x')$  as in (1.8) for each  $x' \in \mathbb{R}^{n-1}$ , define the square function operator  $S_\Theta$  by setting

$$(S_\Theta f)(x') := \left( \int_{\Gamma_\kappa(x')} |(\Theta f)(y', t)|^2 \frac{dy' dt}{t^n} \right)^{\frac{1}{2}}, \quad \forall x' \in \mathbb{R}^{n-1}. \quad (3.42)$$

Then the following are well-defined, linear, and bounded operators:

$$\Theta : L^2(\mathbb{R}^{n-1}) \longrightarrow L^2\left(\mathbb{R}_+^n, \frac{dx' dt}{t}\right), \quad (3.43)$$

$$S_\Theta : L^p(\mathbb{R}^{n-1}) \longrightarrow L^p(\mathbb{R}^{n-1}), \quad \forall p \in (1, 2], \quad (3.44)$$

$$S_\Theta : L^1(\mathbb{R}^{n-1}) \longrightarrow L^{1,\infty}(\mathbb{R}^{n-1}), \quad (3.45)$$

$$S_\Theta : H^1(\mathbb{R}^{n-1}) \longrightarrow L^1(\mathbb{R}^{n-1}). \quad (3.46)$$

*Proof.* We are going to use [6, Theorem 20, p. 69] (see also [7]). First observe that (3.16) with  $\varepsilon = 1$  presently implies

$$|\theta(x', t; y')| \leq C \frac{t}{(t + |x' - y'|)^n} \quad \forall x', y' \in \mathbb{R}^{n-1}, \quad \forall t > 0. \quad (3.47)$$

Second, if  $x', y', z' \in \mathbb{R}^{n-1}$  and  $t > 0$  are such that  $|y' - z'| \leq (t + |x' - y'|)/2$ , the Mean Value Theorem and (3.41) imply (here and elsewhere,  $[a, b]$  denotes the line segment with end-points  $a, b \in \mathbb{R}^{n-1}$ )

$$\begin{aligned} |\theta(x', t; y') - \theta(x', t; z')| &\leq |y' - z'| \sup_{w' \in [y', z']} |\nabla_{y'} \theta(x', t; w')| \\ &\leq C |y' - z'| \sup_{w' \in [y', z']} \frac{t}{(t + |x' - w'|)^{n+1}} \\ &\leq C \frac{|y' - z'| t}{(t + |x' - y'|)^{n+1}}. \end{aligned} \quad (3.48)$$



This proves that the family of kernels  $\{\theta(x', t; y')\}_{t \in (0, \infty)}$  is a standard family in  $\mathbb{R}^{n-1}$  as in [6, Definition 19, p.69]. Third, (3.17) implies that  $\Theta 1(x', t) = 0$  for every  $(x', t) \in \mathbb{R}_+^n$ . We can therefore apply [6, Theorem 20, p.69] to conclude that the operator in (3.43) is well-defined, linear and bounded. In particular, the boundedness of the operator in (3.43) implies that there exists a constant  $C \in (0, \infty)$  such that for every function  $f \in L^2(\mathbb{R}^{n-1})$  there holds

$$\|S_\Theta f\|_{L^2(\mathbb{R}^{n-1})}^2 = C_{n, \kappa} \int_{\mathbb{R}_+^n} |(\Theta f)(y', t)|^2 \frac{dy' dt}{t} \leq C \int_{\mathbb{R}^{n-1}} |f(x')|^2 dx'. \quad (3.49)$$

Proving the boundedness of the operator in (3.45) comes down to establishing the weak-type  $(1, 1)$  estimate for  $S_\Theta$ . In a first stage, we claim that there exists some constant  $C \in (0, \infty)$  with the property that for any cube  $Q$  in  $\mathbb{R}^{n-1}$  and any function  $h$  satisfying

$$h \in L^1(\mathbb{R}^{n-1}), \quad \text{supp } h \subseteq Q, \quad \text{and} \quad \int_{\mathbb{R}^{n-1}} h(y') dy' = 0 \quad (3.50)$$

we have

$$|(S_\Theta h)(x')| \leq C \|h\|_{L^1(\mathbb{R}^{n-1})} \frac{\ell(Q)}{|x' - x'_Q|^n} \quad \text{for every } x' \in \mathbb{R}^{n-1} \setminus 2Q, \quad (3.51)$$

where  $x'_Q$  is the center of the cube  $Q$ . To justify the claim, given  $x', z' \in \mathbb{R}^{n-1}$  with  $x' \neq z'$  we first use (3.41) combined with natural changes of variables to write

$$\begin{aligned} & \int_{\Gamma_\kappa(x')} |\nabla_{y'} \theta(y', t; z')|^2 \frac{dy' dt}{t^n} \\ & \leq C \int_{|y' - x'| < \kappa t} \frac{t^2}{|(y' - z', t)|^{2(n+1)}} \frac{dy' dt}{t^n} \\ & = C \int_0^\infty \int_{|y'| < \kappa} \frac{t^2}{|(y't + (x' - z'), t)|^{2(n+1)}} \frac{dy' dt}{t} \\ & = C \int_0^\infty \int_{|y'| < \kappa} \frac{t^{-2n}}{|(y' + (x' - z')/t, 1)|^{2(n+1)}} \frac{dy' dt}{t} \\ & \leq C |x' - z'|^{-2n} \int_0^\infty \int_{|y'| < \kappa} \frac{t^{-2n}}{|(y' + (x' - z')/(t|x' - z'|), 1)|^{2(n+1)}} \frac{dy' dt}{t} \\ & \leq C |x' - z'|^{-2n} \sup_{|v'|=1} \int_0^\infty \int_{|y'| < \kappa} \frac{t^{-2n}}{|y' + v'/t|^{2(n+1)} + 1} \frac{dy' dt}{t}. \end{aligned} \quad (3.52)$$

Next, fix  $v' \in \mathbb{R}^{n-1}$  with  $|v'| = 1$ . If  $t < 1/(2\kappa)$  and  $|y'| < \kappa$  we have

$$t^{-1} = |v'|/t \leq |y' + v'/t| + |y'| < |y' + v'/t| + 1/(2t) \quad (3.53)$$

and therefore  $|y' + v'/t| > 1/(2t)$ . Thus,

$$\int_0^{1/(2\kappa)} \int_{|y'| < \kappa} \frac{t^{-2n}}{|y' + v'/t|^{2(n+1)} + 1} \frac{dy' dt}{t} \leq C \int_0^{1/(2\kappa)} \int_{|y'| < \kappa} t dy' dt \leq C. \quad (3.54)$$

Also, it is immediate that

$$\int_{1/(2\kappa)}^\infty \int_{|y'| < \kappa} \frac{t^{-2n}}{|y' + v'/t|^{2(n+1)} + 1} \frac{dy' dt}{t} \leq C \int_{1/(2\kappa)}^\infty \int_{|y'| < \kappa} t^{-2n} \frac{dy' dt}{t} \leq C. \quad (3.55)$$

Combining (3.52), (3.54), and (3.55) we may conclude that

$$\left( \int_{\Gamma_\kappa(x')} |\nabla_{y'} \theta(y', t; z')|^2 \frac{dy' dt}{t^n} \right)^{\frac{1}{2}} \leq \frac{C}{|x' - z'|^n}, \quad \text{if } x' \neq z'. \quad (3.56)$$

At this point we return to the proof of (3.51). Fix  $x' \in \mathbb{R}^{n-1} \setminus 2Q$  and consider  $h$  as in (3.50). Making use of the last property of  $h$  recorded in (3.50), the Fundamental Theorem of Calculus, Minkowski's inequality, and (3.56) we may compute

$$\begin{aligned} (S_\Theta h)(x') &= \left( \int_{\Gamma_\kappa(x')} |(\Theta h)(y', t)|^2 \frac{dy' dt}{t^n} \right)^{\frac{1}{2}} \\ &\leq \left( \int_{\Gamma_\kappa(x')} \left( \int_{\mathbb{R}^{n-1}} |\theta(y', t; z') - \theta(y', t; x'_Q)| |h(z')| dz' \right)^2 \frac{dy' dt}{t^n} \right)^{\frac{1}{2}} \\ &\leq \int_0^1 \int_{\mathbb{R}^{n-1}} \left( \int_{\Gamma_\kappa(x')} |\nabla_{z'} \theta(y', t; x'_Q + s(z' - x'_Q))|^2 \frac{dy' dt}{t^n} \right)^{\frac{1}{2}} |h(z')| |z' - x'_Q| dz' ds \\ &\leq C \int_0^1 \int_Q \frac{1}{|x' - (x'_Q + s(z' - x'_Q))|^n} |h(z')| |z' - x'_Q| dz' ds \\ &\leq C \|h\|_{L^1(\mathbb{R}^{n-1})} \frac{\ell(Q)}{|x' - x'_Q|^n}. \end{aligned} \quad (3.57)$$

For the last inequality in (3.57) we used the observation that for every  $s \in (0, 1)$  and every  $z' \in Q$  one has (keeping in mind that  $x' \in \mathbb{R}^{n-1} \setminus 2Q$  and  $x'_Q + s(z' - x'_Q) \in Q$ )

$$\begin{aligned} |x' - x'_Q| &\leq |x' - (x'_Q + s(z' - x'_Q))| + \frac{\sqrt{n-1} \ell(Q)}{2} \\ &\leq |x' - (x'_Q + s(z' - x'_Q))| + \sqrt{n-1} |x' - (x'_Q + s(z' - x'_Q))| \\ &= (1 + \sqrt{n-1}) |x' - (x'_Q + s(z' - x'_Q))|. \end{aligned} \quad (3.58)$$

This finishes the proof of (3.51).

Let us momentarily digress to show that

$$\int_{\mathbb{R}^{n-1} \setminus Q} \frac{\ell(Q)}{|x' - x'_Q|^n} dx' \leq \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} \frac{\ell(Q)}{(\ell(2^k Q)/2)^n} dx' \leq 2^{2n-1} \sum_{k=0}^{\infty} 2^{-k} = 4^n. \quad (3.59)$$

We are now ready to show that  $S_\Theta$  maps  $L^1(\mathbb{R}^{n-1})$  continuously into  $L^{1,\infty}(\mathbb{R}^{n-1})$ . Following [14, p.140], given a function  $f \in L^1(\mathbb{R}^{n-1})$  and fixed  $\lambda > 0$ , let  $\{Q_j\}_j \subset \mathbb{R}^{n-1}$  be the non-overlapping cubes of the Calderón-Zygmund decomposition of  $|f|$  at height  $\lambda$ . That is, the  $Q_j$ 's are the maximal dyadic cubes for which  $|Q_j|^{-1} \int_{Q_j} |f(y')| dy' > \lambda$ . Set

$$\Omega_\lambda = \bigcup_{j=1}^{\infty} Q_j \quad (3.60)$$

and observe that we have the following properties

$$\mathcal{L}^{n-1}(\Omega_\lambda) \leq \lambda^{-1} \|f\|_{L^1(\mathbb{R}^{n-1})}, \quad (3.61)$$

$$\lambda < \oint_{Q_j} |f(y')| dy' \leq 2^{n-1} \lambda, \quad \forall j \in \mathbb{N}, \quad (3.62)$$

and

$$|f(x')| \leq \lambda, \quad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \mathbb{R}^{n-1} \setminus \Omega_\lambda. \quad (3.63)$$

Finally, split  $f = g + b$  where (cf. [14, p. 198])

$$g := f \mathbf{1}_{\mathbb{R}^{n-1} \setminus \Omega_\lambda} + \sum_{j=1}^{\infty} f_{Q_j} \mathbf{1}_{Q_j}, \quad b = \sum_{j=1}^{\infty} b_j, \quad (3.64)$$

with  $b_j := (f - f_{Q_j}) \mathbf{1}_{Q_j}$  for each  $j \in \mathbb{N}$ .

In particular, (3.60)-(3.64) imply (cf. [14, p. 198]) that for some constant  $C \in (0, \infty)$  independent of  $f$  and  $\lambda$  we have

$$\|g\|_{L^2(\mathbb{R}^{n-1})}^2 \leq 2^{n-1} \lambda \|f\|_{L^1(\mathbb{R}^{n-1})}. \quad (3.65)$$

Making use of (3.64), (3.61), (3.65), (3.49)-(3.51), (3.59) (used with  $Q_j$  in place of  $Q$ ), and bearing in mind that for each  $j \in \mathbb{N}$  we have  $\text{supp } b_j \subset Q_j$  and  $\int_{\mathbb{R}^{n-1}} b_j(y') dy' = 0$ , we may then estimate

$$\begin{aligned} & \lambda \mathcal{L}^{n-1}(\{x' \in \mathbb{R}^{n-1} : (S_\Theta f)(x') > \lambda\}) \\ & \leq \lambda \mathcal{L}^{n-1}(\{x' \in \mathbb{R}^{n-1} : (S_\Theta g)(x') > \lambda/2\}) \\ & \quad + \lambda \mathcal{L}^{n-1}(\{x' \in \mathbb{R}^{n-1} : (S_\Theta b)(x') > \lambda/2\}) \\ & \leq \frac{4}{\lambda} \int_{\mathbb{R}^{n-1}} |(S_\Theta g)(x')|^2 dx' + \lambda \mathcal{L}^{n-1}\left(\bigcup_{j=1}^{\infty} 2Q_j\right) \\ & \quad + 2 \sum_{j=1}^{\infty} \int_{\mathbb{R}^{n-1} \setminus 2Q_j} |(S_\Theta b_j)(x')| dx' \\ & \leq \frac{C}{\lambda} \int_{\mathbb{R}^{n-1}} |g(x')|^2 dx' + C \|f\|_{L^1(\mathbb{R}^{n-1})} \\ & \quad + C \sum_{j=1}^{\infty} \|b_j\|_{L^1(\mathbb{R}^{n-1})} \int_{\mathbb{R}^{n-1} \setminus 2Q_j} \frac{\ell(Q_j)}{|x' - x'_{Q_j}|^n} dx' \\ & \leq C \|f\|_{L^1(\mathbb{R}^{n-1})} + C \sum_{j=1}^{\infty} \|b_j\|_{L^1(\mathbb{R}^{n-1})} \\ & \leq C \|f\|_{L^1(\mathbb{R}^{n-1})} + C \left( \sum_{j=1}^{\infty} \int_{Q_j} |f(y')| dy' \right) \\ & \leq C \|f\|_{L^1(\mathbb{R}^{n-1})}. \end{aligned} \quad (3.66)$$

This proves that the operator in (3.45) is well-defined, linear and bounded. The latter and Marcinkewicz's Interpolation Theorem then imply the boundedness of the operator in (3.44).

Next we consider  $S_\Theta$  in the context of (3.46). In this regard, we shall first show that there exists some constant  $C \in (0, \infty)$  such that for every  $(1, \infty)$ -atom  $a$  one has

$$\|S_\Theta a\|_{L^1(\mathbb{R}^{n-1})} \leq C. \quad (3.67)$$

To justify (3.67) fix an arbitrary function  $a$  satisfying the conditions listed in (3.37). On the one hand, based on Hölder's inequality, (3.49), and the first two properties in (3.37) we may write

$$\int_{2Q} |(S_\Theta a)(x')| dx' \leq C |Q|^{\frac{1}{2}} \|S_\Theta a\|_{L^2(\mathbb{R}^{n-1})} \leq C |Q|^{\frac{1}{2}} \|a\|_{L^2(\mathbb{R}^{n-1})} \leq C, \quad (3.68)$$

for some finite constant  $C > 0$  independent of  $a$ . On the other hand, (3.37) allows us to make use of (3.51) (with  $a$  in place of  $h$ ), which we combine with the second property in (3.37) and (3.59) to obtain

$$\int_{\mathbb{R}^{n-1} \setminus 2Q} |(S_{\Theta} a)(x')| dx' \leq C \|a\|_{L^1(\mathbb{R}^{n-1})} \int_{\mathbb{R}^{n-1} \setminus 2Q} \frac{\ell(Q)}{|x' - x'_Q|^n} dx' \leq C, \quad (3.69)$$

with  $C \in (0, \infty)$  independent of the atom  $a$ . Combining (3.68) and (3.69) then proves that (3.67) holds.

Here is the end-game in the proof of the fact that  $S_{\Theta}$  maps  $H^1(\mathbb{R}^{n-1})$  boundedly into  $L^1(\mathbb{R}^{n-1})$ . Let  $f \in H^1(\mathbb{R}^{n-1})$  be arbitrary and consider an atomic decomposition  $f = \sum_{j=1}^{\infty} \lambda_j a_j$  convergent in  $H^1(\mathbb{R}^{n-1})$ , where the  $a_j$ 's are  $(1, \infty)$ -atoms, which is quasi-optimal in the sense that  $\sum_{j=1}^{\infty} |\lambda_j| \approx \|f\|_{H^1(\mathbb{R}^{n-1})}$ , where the proportionality constants do not depend on  $f$ . In particular, this forces  $f = \sum_{j=1}^{\infty} \lambda_j a_j$  in  $L^1(\mathbb{R}^{n-1})$  and the weak-type  $(1, 1)$  estimate for  $S_{\Theta}$  then implies  $S_{\Theta} f = \sum_{j=1}^{\infty} \lambda_j S_{\Theta} a_j$  in  $L^{1,\infty}(\mathbb{R}^{n-1})$ . Then the sequence of partial sums associated with the latter series has a sub-sequence which converges a.e. to  $S_{\Theta} f$ . In turn, this allows us to conclude that

$$|(S_{\Theta} f)(x')| \leq \sum_{j=1}^{\infty} |\lambda_j| |(S_{\Theta} a_j)(x')| \quad \text{for a.e. } x' \in \mathbb{R}^{n-1}. \quad (3.70)$$

In concert, (3.70) and (3.67) then imply

$$\|S_{\Theta} f\|_{L^1(\mathbb{R}^{n-1})} \leq \sum_{j=1}^{\infty} |\lambda_j| \|S_{\Theta} a_j\|_{L^1(\mathbb{R}^{n-1})} \leq C \sum_{j=1}^{\infty} |\lambda_j| \leq C \|f\|_{H^1(\mathbb{R}^{n-1})}, \quad (3.71)$$

as desired, for some constant  $C \in (0, \infty)$  independent of  $f$ .  $\square$

We now have all the ingredients to proceed with the proof of Proposition 3.1.

*Proof of Proposition 3.1.* Fix an arbitrary  $f \in L^1\left(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n}\right)^M$  and define  $u$  as in (3.3). Part (7) in Theorem 2.3 then ensures that this function satisfies all properties listed in (3.4).

As in (2.37), write  $K^L(x', t) = P_t^L(x')$  for each  $(x', t) \in \mathbb{R}_+^n$ . To proceed, fix an arbitrary point  $(x', t) \in \mathbb{R}_+^n$  and denote by  $Q_{x',t}$  the cube in  $\mathbb{R}^{n-1}$  centered at  $x'$  with side-length  $t$ . Making use of (2.36) we obtain

$$\int_{\mathbb{R}^{n-1}} \nabla^\ell [P_t^L(x' - y')] dy' = 0, \quad \forall x' \in \mathbb{R}^{n-1}, \forall t > 0, \forall \ell \in \mathbb{N}. \quad (3.72)$$

Based on this, for each  $\ell \in \mathbb{N}$  we may then write

$$\begin{aligned} (\nabla^\ell u)(x', t) &= \int_{\mathbb{R}^{n-1}} \nabla^\ell [P_t^L(x' - y')] f(y') dy' \\ &= \int_{\mathbb{R}^{n-1}} \nabla^\ell [P_t^L(x' - y')] [f(y') - f_{Q_{x',t}}] dy' \\ &= \int_{\mathbb{R}^{n-1}} (\nabla^\ell K^L)(x' - y', t) [f(y') - f_{Q_{x',t}}] dy'. \end{aligned} \quad (3.73)$$

Combining (3.73), (2.39), and (2.21) (with  $\varepsilon = \ell$ ), we may now estimate

$$|(\nabla^\ell u)(x', t)| \leq C \int_{\mathbb{R}^{n-1}} \frac{|f(y') - f_{Q_{x',t}}|}{|(x' - y', t)|^{n-1+\ell}} dy' \leq \frac{C}{t^\ell} \int_1^\infty \text{osc}_1(f; \lambda t) \frac{d\lambda}{\lambda^{1+\ell}}, \quad (3.74)$$

from which the claims in part (a) of the statement follow.

Moving on, fix an arbitrary  $j \in \{1, \dots, n\}$  and, for each  $\alpha, \beta \in \{1, \dots, M\}$ , set

$$\theta_{\alpha\beta}^j(x', t; y') := t \partial_j K_{\alpha\beta}^L(x' - y', t), \quad \forall x', y' \in \mathbb{R}_+^{n-1}, \quad \forall t > 0. \quad (3.75)$$

In this regard, first observe that (2.39) in Theorem 2.3 implies

$$|\theta_{\alpha\beta}^j(x', t; y')| = t |\partial_j K_{\alpha\beta}^L(x' - y', t)| \leq C t |(x' - y', t)|^{-n} \quad (3.76)$$

and

$$|\nabla_{y'} \theta_{\alpha\beta}^j(x', t; y')| \leq t |\nabla^2 K_{\alpha\beta}^L(x' - y', t)| \leq C t |(x' - y', t)|^{-n-1}. \quad (3.77)$$

Hence, (3.16) (with  $\varepsilon = 1$ ) and (3.41) hold for  $\theta_{\alpha\beta}^j$ . Moreover,

$$\int_{\mathbb{R}^{n-1}} \theta_{\alpha\beta}^j(x', t; y') dy' = \int_{\mathbb{R}^{n-1}} t \partial_j K_{\alpha\beta}^L(x' - y', t) dy' = t \partial_j \int_{\mathbb{R}^{n-1}} K_{\alpha\beta}^L(y', t) dy' = 0 \quad (3.78)$$

since  $\partial_j \int_{\mathbb{R}^{n-1}} (P_{\alpha\beta}^L)_t(y') dy' = 0$  by (3.72). Writing  $\Theta_{\alpha\beta}^j$  for the operator associated with the kernel  $\theta_{\alpha\beta}^j$  (in place of  $\theta$ ) as in (3.18), it follows from (3.76), (3.77), (3.78), and Proposition 3.3 that each matrix integral operator  $\Theta^j := (\Theta_{\alpha\beta}^j)_{1 \leq \alpha, \beta \leq M}$  satisfies all hypotheses in Proposition 3.2 (including (3.19)). In addition,

$$\begin{aligned} \left( \int_0^{\ell(Q)} \int_Q |\nabla u(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} &= \left( \int_0^{\ell(Q)} \int_Q |t \nabla (P_t^L * f)(x')|^2 \frac{dx' dt}{t} \right)^{\frac{1}{2}} \\ &\leq \sum_{j=1}^n \left( \int_0^{\ell(Q)} \int_Q |t (\partial_j K^L(\cdot, t) * f)(x')|^2 \frac{dx' dt}{t} \right)^{\frac{1}{2}} \\ &= \sum_{j=1}^n \left( \int_0^{\ell(Q)} \int_Q |(\Theta^j f)(x', t)|^2 \frac{dx' dt}{t} \right)^{\frac{1}{2}}. \end{aligned} \quad (3.79)$$

Granted this, all remaining conclusions in parts (b)-(f) of the statement become direct consequences of Proposition 3.2.  $\square$

#### 4. A FATOU RESULT AND UNIQUENESS IN THE BMO-DIRICHLET PROBLEM

The main result in this section is the following Poisson representation formula and Fatou theorem.

**Proposition 4.1.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2). Assume that*

$$u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M), \quad Lu = 0 \quad \text{in } \mathbb{R}_+^n, \quad \text{and} \quad \|u\|_{**} < \infty. \quad (4.1)$$

*Then there exists a unique function  $f \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  such that*

$$u(x', t) = (P_t^L * f)(x'), \quad \forall (x', t) \in \mathbb{R}_+^n, \quad (4.2)$$

*where  $P^L$  is the Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.3.*

In fact,  $u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}}$  exists at a.e. point in  $\mathbb{R}^{n-1}$  and  $f = u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}}$ . Moreover, there exists a constant  $C = C(n, L) \in (1, \infty)$  such that

$$C^{-1}\|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq \|u\|_{**} \leq C\|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)}. \quad (4.3)$$

Also, as a corollary of Proposition 4.1 we have the following result which, in view of (1.13), implies the uniqueness statements for the BMO-Dirichlet problem and the VMO-Dirichlet problem from Theorem 1.1.

**Proposition 4.2.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2). Assume that*

$$\begin{aligned} u &\in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M), \quad Lu = 0 \quad \text{in } \mathbb{R}_+^n, \quad \|u\|_{**} < \infty, \\ u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} &\text{ exists and vanishes at a.e. point in } \mathbb{R}^{n-1}. \end{aligned} \quad (4.4)$$

Then necessarily  $u \equiv 0$  in  $\mathbb{R}_+^n$ .

The proof of Proposition 4.1 occupies the bulk of this section. To set the stage, we first prove some auxiliary lemmas. The first such lemma contains Bloch-like estimates for smooth null-solutions of  $L$  satisfying a Carleson measure condition in the upper-half space. To place things in perspective, recall that a holomorphic function  $u$  in the upper-half plane is said to satisfy a Bloch estimate provided

$$\sup_{x \in \mathbb{R}, y > 0} (y|u'(x + iy)|) < \infty. \quad (4.5)$$

**Lemma 4.3.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2). Then for every multi-index  $\alpha \in \mathbb{N}_0^n$  with  $|\alpha| \geq 1$  there exists a finite constant  $C = C(n, L, \alpha) > 0$  with the property that for every function  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$  satisfying  $Lu = 0$  in  $\mathbb{R}_+^n$  and  $\|u\|_{**} < \infty$  one has*

$$\sup_{(x', t) \in \mathbb{R}_+^n} \left\{ t^{|\alpha|} |(\partial^\alpha u)(x', t)| \right\} \leq C \|u\|_{**}. \quad (4.6)$$

In particular, there exists a finite constant  $C = C(n, L) > 0$  with the property that for every function  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$  such that  $Lu = 0$  in  $\mathbb{R}_+^n$  and  $\|u\|_{**} < \infty$  one has

$$\sup_{(x', t) \in \mathbb{R}_+^n} t |\nabla u(x', t)| \leq C \|u\|_{**}. \quad (4.7)$$

*Proof.* Given a multi-index  $\alpha \in \mathbb{N}_0^n$  with  $|\alpha| \geq 1$ , select  $j \in \{1, \dots, n\}$  and  $\beta \in \mathbb{N}_0^n$  such that  $\alpha = \beta + e_j$ . Fix  $x = (x', t) \in \mathbb{R}_+^n$  and write  $R_x$  for the cube in  $\mathbb{R}^n$  centered at  $x$  with side-length  $t$ . Also, let  $Q_{x'}$  be the cube in  $\mathbb{R}^{n-1}$  centered at  $x'$  with side-length  $t$ . Since the function  $\partial_j u$  is a null-solution of the system  $L$ , we may invoke Theorem 2.4 (with  $\partial_j u$  in place of  $u$  and  $p = 2$ ) in order to conclude

$$|\partial^\beta(\partial_j u)(x', t)| \leq \frac{C_\beta}{t^{|\beta|}} \left( \int_{R_x} |\partial_j u|^2 d\mathcal{L}^n \right)^{\frac{1}{2}}. \quad (4.8)$$

Hence,

$$\begin{aligned} t^{|\alpha|} |(\partial^\alpha u)(x', t)| &\leq C t \left( \int_{R_x} |\nabla u|^2 d\mathcal{L}^n \right)^{\frac{1}{2}} \\ &\leq C \left( \int_{t/2}^{3t/2} \int_{Q_{x'}} |\nabla u(y', s)|^2 s dy' ds \right)^{\frac{1}{2}} \\ &\leq C \|u\|_{**}, \end{aligned} \quad (4.9)$$



proving (4.6). Estimate (4.7) is a particular case of (4.6).  $\square$

We continue by discussing two purely real-variable results. To state the first one, recall the function  $\Upsilon_{\#} : [0, \infty) \rightarrow [0, \infty)$  from (1.44). In relation to this, we make two observations. First,

$$\begin{aligned} &\text{for each } \varepsilon \in (0, \infty) \text{ there exists } C_{\varepsilon} \in (1, \infty) \text{ such that} \\ &C_{\varepsilon}^{-1} \Upsilon_{\#}(s) \leq \Upsilon_{\#}(s/\varepsilon) \leq C_{\varepsilon} \Upsilon_{\#}(s) \text{ for every } s \in [0, \infty). \end{aligned} \quad (4.10)$$

Second, since for every  $\eta \in (0, 1]$  there exists a constant  $C = C_{\eta} \in (0, \infty)$  with the property that

$$\Upsilon_{\#}(s) \leq C s^{\eta}, \quad \forall s \geq 0, \quad (4.11)$$

we have (cf. (1.42))

$$\mathcal{C}^{\Upsilon_{\#}}(\mathbb{R}^{n-1}, \mathbb{C}^M) \subset \text{Lip}(\mathbb{R}^{n-1}, \mathbb{C}^M) \cap \left( \bigcap_{0 < \eta < 1} \mathcal{C}^{\eta}(\mathbb{R}^{n-1}, \mathbb{C}^M) \right). \quad (4.12)$$

This is going to be relevant later on, in the proof of Lemma 4.6. For now, here is the first real-variable result advertised above.

**Lemma 4.4.** *Recall  $\Upsilon_{\#}$  from (1.44) and let  $u \in \mathcal{C}^1(\mathbb{R}_+^n, \mathbb{C}^M)$  be such that*

$$C_u := \sup_{(x', t) \in \mathbb{R}_+^n} t |\nabla u(x', t)| < \infty. \quad (4.13)$$

*Then, for every  $(x', t)$  and  $(y', t)$  in  $\mathbb{R}_+^n$  one has*

$$|u(x', t) - u(y', t)| \leq 2 C_u \Upsilon_{\#}\left(\frac{|x' - y'|}{t}\right). \quad (4.14)$$

*Proof.* The proof follows the argument in [11]. Fix  $(x', t)$  and  $(y', t)$  in  $\mathbb{R}_+^n$ . Based on the Mean Value Theorem and (4.13) we may estimate

$$|u(x', t) - u(y', t)| \leq \sup_{\xi \in [x', y']} |\nabla u(\xi, t)| |x' - y'| \leq C_u \frac{|x' - y'|}{t}. \quad (4.15)$$

Suppose now that  $|x' - y'| > t$  and set  $r := |x' - y'|$ . Applying the Fundamental Theorem of Calculus, (4.15), and (4.13) we obtain

$$\begin{aligned} |u(x', t) - u(y', t)| &\leq |u(x', t) - u(x', r)| + |u(x', r) - u(y', r)| + |u(y', r) - u(y', t)| \\ &\leq \int_t^r |\partial_n u(x', \lambda)| d\lambda + C_u + \int_t^r |\partial_n u(y', \lambda)| d\lambda \\ &\leq C_u + 2 C_u \int_t^r \frac{1}{\lambda} d\lambda \\ &\leq 2 C_u \left(1 + \ln \frac{|x' - y'|}{t}\right). \end{aligned} \quad (4.16)$$

With this in hand, (4.14) follows from (4.16) (which is valid for  $|x' - y'| > t$ ) and (4.15) used for  $|x' - y'| \leq t$ .  $\square$

The second real-variable result mentioned earlier reads as follows.

**Lemma 4.5.** *Let  $\Upsilon_{\#}$  be the function defined in (1.44). Then for every  $a > 0$  one has*

$$\Psi(a) := \int_0^\infty \frac{s^{n-1}}{(a+s)^n} \Upsilon_{\#}(s) \frac{ds}{s} \leq 3 \begin{cases} 1 + \ln \frac{1}{a}, & \text{if } a \leq 1, \\ \frac{1 + \ln a}{a}, & \text{if } a > 1. \end{cases} \quad (4.17)$$

*In particular,  $\Psi(a) \leq 3 \left(1 + \log^+ \frac{1}{a}\right)$ , where  $\log^+ s := \max\{\ln s, 0\}$  for every  $s \in (0, \infty)$ .*

*Proof.* If  $a \geq 1$ , we use that  $s^{n-2} \Upsilon_{\#}(s)$  is increasing and elementary calculus to obtain

$$\begin{aligned} \Psi(a) &\leq a^{-n} \int_0^a s^{n-2} \Upsilon_{\#}(s) ds + \int_a^\infty \frac{1 + \ln s}{s^2} ds \\ &\leq a^{-n} a^{n-2} \Upsilon_{\#}(a) a + a^{-1} + \left[ \frac{-1 - \ln s}{s} \right]_{s=a}^{s=\infty} \leq 3 \frac{1 + \ln a}{a}. \end{aligned} \quad (4.18)$$

On the other hand, if  $a < 1$  then

$$\begin{aligned} \Psi(a) &\leq \int_0^a \frac{s^{n-1}}{a^n} s \frac{ds}{s} + \int_a^1 \frac{s^{n-1}}{s^n} s \frac{ds}{s} + \int_1^\infty \frac{s^{n-1}}{s^n} (1 + \ln s) \frac{ds}{s} \\ &= \frac{1}{n} + \ln \frac{1}{a} + 2 \leq 3 \left( 1 + \ln \frac{1}{a} \right). \end{aligned} \quad (4.19)$$

Collectively, (4.18)-(4.19) prove the lemma.  $\square$

Having dealt with Lemmas 4.4-4.5, in our next two lemmas we study the boundary behavior of the vertical shifts of a smooth null-solution of  $L$  which satisfies a Carleson measure condition in the upper-half space.

**Lemma 4.6.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2) and consider  $P^L$ , the associated Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.3. Suppose  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$  satisfies  $Lu = 0$  in  $\mathbb{R}_+^n$  and  $\|u\|_{**} < \infty$ . For each  $\varepsilon > 0$  define  $u_\varepsilon(x', t) := u(x', t + \varepsilon)$ , for every  $(x', t) \in \mathbb{R}_+^n$  and  $f_\varepsilon(x') := u(x', \varepsilon)$  for every  $x' \in \mathbb{R}^{n-1}$ . Then there exists a constant  $C \in (0, \infty)$  such that for every  $\varepsilon > 0$  the following properties are valid:*

- (a) *The function  $u_\varepsilon$  belongs to  $\mathcal{C}^\infty(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)$  and  $Lu_\varepsilon = 0$  in  $\mathbb{R}_+^n$ .*
- (b) *One has  $\|u_\varepsilon\|_{**} \leq C \|u\|_{**}$ . In fact, for every multi-index  $\alpha \in \mathbb{N}_0^n$  there exists a constant  $C_\alpha \in (0, \infty)$ , independent of  $u$  and  $\varepsilon$ , with the property that  $\|\partial^\alpha u_\varepsilon\|_{**} \leq C_\alpha \varepsilon^{-|\alpha|} \|u\|_{**}$ .*
- (c) *For every multi-index  $\alpha \in \mathbb{N}_0^n$  with  $|\alpha| \geq 1$  there exists a constant  $C_\alpha \in (0, \infty)$ , independent of  $u$ , with the property that  $\|\partial^\alpha u_\varepsilon\|_{L^\infty(\mathbb{R}_+^n)} \leq C_\alpha \varepsilon^{-|\alpha|} \|u\|_{**}$ .*
- (d) *The function  $f_\varepsilon$  belongs to  $\mathcal{C}^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M) \cap \mathcal{C}^{\Upsilon_{\#}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  where  $\Upsilon_{\#}$  is as in (1.44). In particular,*

$$f_\varepsilon \in \text{Lip}(\mathbb{R}^{n-1}, \mathbb{C}^M) \cap \left( \bigcap_{0 < \eta < 1} \mathcal{C}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M) \right) \quad (4.20)$$

hence also  $f_\varepsilon \in \text{UC}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  and

$$f_\varepsilon \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1 + |x'|^n} dx'\right)^M. \quad (4.21)$$

Moreover,

$$\begin{aligned} &\text{for every } \alpha' \in \mathbb{N}_0^{n-1} \text{ with } |\alpha'| \geq 1 \text{ one has} \\ &\partial^{\alpha'} f_\varepsilon \in L^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M) \cap \mathcal{C}^{\Upsilon_{\#}}(\mathbb{R}^{n-1}, \mathbb{C}^M). \end{aligned} \quad (4.22)$$

- (e) *The function  $v_\varepsilon(x', t) := (P_t^L * f_\varepsilon)(x')$  is well-defined for all  $(x', t) \in \mathbb{R}_+^n$  via an absolutely convergent integral and*

$$v_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M), \quad Lv_\varepsilon = 0 \text{ in } \mathbb{R}_+^n, \quad v_\varepsilon|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} = f_\varepsilon \text{ everywhere in } \mathbb{R}^{n-1}. \quad (4.23)$$

- (f) *For every  $(y', t) \in \mathbb{R}_+^n$  one has*

$$|v_\varepsilon(y', t) - f_\varepsilon(y')| + t |\nabla v_\varepsilon(y', t)| \leq C \|u\|_{**} (t/\varepsilon) (1 + \log^+(\varepsilon/t)). \quad (4.24)$$

*Proof.* The claim in part (a) is clear from definitions. To prove the estimate in part (b), fix a cube  $Q \subset \mathbb{R}^{n-1}$ . Consider first the case  $\ell(Q) \geq \varepsilon$  in which scenario a change of variables yields

$$\begin{aligned} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\nabla u(x', t + \varepsilon)|^2 t \, dx' \, dt &\leq \frac{1}{|Q|} \int_\varepsilon^{\ell(Q) + \varepsilon} \int_Q |\nabla u(x', t)|^2 t \, dx' \, dt \\ &\leq 2^{n-1} \frac{1}{|2Q|} \int_0^{2\ell(Q)} \int_{2Q} |\nabla u(x', t)|^2 t \, dx' \, dt \\ &\leq 2^{n-1} \|u\|_{**}^2. \end{aligned} \quad (4.25)$$

In the case  $\ell(Q) < \varepsilon$ , use Lemma 4.3 to conclude that

$$\begin{aligned} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\nabla u(x', t + \varepsilon)|^2 t \, dx' \, dt &\leq C^2 \|u\|_{**}^2 \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q \frac{1}{(t + \varepsilon)^2} t \, dx' \, dt \\ &\leq C^2 \|u\|_{**}^2 \varepsilon^{-2} \int_0^{\ell(Q)} t \, dt \leq \frac{C^2}{2} \|u\|_{**}^2, \end{aligned} \quad (4.26)$$

for some  $C \in (0, \infty)$  independent of  $u$  and  $\varepsilon$ . Combining (4.25) and (4.26) and taking the supremum over all cubes  $Q$  then proves the first estimate in part (b) for some  $C \in (0, \infty)$  independent of  $u$  and  $\varepsilon$ .

To justify the second estimate in part (b), it suffices to consider the case when the multi-index  $\alpha \in \mathbb{N}_0^n$  has length  $|\alpha| \geq 1$ . Assume that this is the case and pick an arbitrary cube  $Q \subset \mathbb{R}^{n-1}$ . Making use of (4.6) and bearing in mind that  $|\alpha| \geq 1$  we may then estimate

$$\begin{aligned} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\nabla[(\partial^\alpha u_\varepsilon)(x', t)]|^2 t \, dx' \, dt &= \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\nabla[(\partial^\alpha u)(x', t + \varepsilon)]|^2 t \, dx' \, dt \\ &\leq \frac{C_\alpha \|u\|_{**}^2}{|Q|} \int_0^{\ell(Q)} \int_Q \frac{1}{(t + \varepsilon)^{2|\alpha|+1}} \, dx' \, dt \\ &\leq C_\alpha \|u\|_{**}^2 \int_0^\infty \frac{1}{(t + \varepsilon)^{2|\alpha|+1}} \, dt \\ &\leq C_\alpha \|u\|_{**}^2 \varepsilon^{-2|\alpha|}, \end{aligned} \quad (4.27)$$

from which the desired conclusion readily follows.

Consider next the claim in part (c). Given a multi-index  $\alpha \in \mathbb{N}_0^n$  with  $|\alpha| \geq 1$  we may invoke Lemma 4.3 (keeping in mind the conclusions in part (a)) in order to conclude that there exists a constant  $C_\alpha \in (0, \infty)$ , independent of  $u$ , such that

$$\begin{aligned} \sup_{(x', t) \in \mathbb{R}_+^n} |(\partial^\alpha u_\varepsilon)(x', t)| &\leq \sup_{(x', t) \in \mathbb{R}_+^n} \left[ (t + \varepsilon)^{|\alpha|} |(\partial^\alpha u)(x', t + \varepsilon)| \right] \cdot \sup_{(x', t) \in \mathbb{R}_+^n} (t + \varepsilon)^{-|\alpha|} \\ &\leq C_\alpha \|u\|_{**} \varepsilon^{-|\alpha|}. \end{aligned} \quad (4.28)$$

We now turn to proving the claims in part (d). First, since  $f_\varepsilon(y') = u(y', \varepsilon)$  for every  $y' \in \mathbb{R}^{n-1}$  we have  $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . Second, by using (4.14) (with  $t = \varepsilon$ ), (4.7), and (4.10), for each  $x', y' \in \mathbb{R}^{n-1}$  we may estimate

$$|f_\varepsilon(x') - f_\varepsilon(y')| \leq C \|u\|_{**} \Upsilon_\# \left( \frac{|x' - y'|}{\varepsilon} \right) \leq C_{n, L, u, \varepsilon} \Upsilon_\#(|x' - y'|). \quad (4.29)$$

This places  $f_\varepsilon$  in  $\mathcal{C}^{\Upsilon\#}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . With this in hand, the conclusions in (4.20) follow with the help of (4.12). As a Hölder function,  $f_\varepsilon$  also belongs to  $L^1\left(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n} dx'\right)^M$  (see (2.26)), proving (4.21).

Next, fix a multi-index  $\alpha' \in \mathbb{N}_0^{n-1}$  of length  $|\alpha'| \geq 1$ . Then  $\partial^{(\alpha',0)} u_\varepsilon \in \mathcal{C}^\infty(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)$  is a null-solution of  $L$  in  $\mathbb{R}_+^n$  and  $\partial^{\alpha'} f_\varepsilon = (\partial^{(\alpha',0)} u_\varepsilon)|_{\partial\mathbb{R}_+^n}$ . Now the claim in (4.22) is a consequence of parts (c) and (b), bearing in mind that  $\|\partial^{(\alpha',0)} u_\varepsilon\|_{**} < \infty$  (hence, the same type of argument that placed  $f_\varepsilon$  in  $\mathcal{C}^{\Upsilon\#}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  now ensures the membership of  $\partial^{\alpha'} f_\varepsilon$  to the latter space).

Moving on, the claim made in part (e) is a consequence of the current part (d) together with part (7) in Theorem 2.3 and the fact that since  $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M)$  all points in  $\mathbb{R}^{n-1}$  are Lebesgue points for  $f_\varepsilon$ .

Finally, consider the claim in part (f). Fix  $(y', t) \in \mathbb{R}_+^n$ . Then the properties of the Poisson kernel recalled in Theorem 2.3, together with Lemmas 4.3, 4.4, and 4.5 permit us to estimate (bearing in mind that  $f_\varepsilon = u(\cdot, \varepsilon)$ )

$$\begin{aligned}
|v_\varepsilon(y', t) - f_\varepsilon(y')| &\leq \int_{\mathbb{R}^{n-1}} |P^L(z')| |f_\varepsilon(y' - tz') - f_\varepsilon(y')| dz' \\
&\leq C \|u\|_{**} \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|z'|)^n} \Upsilon_{\#}\left(\frac{t|z'|}{\varepsilon}\right) dz' \\
&\leq C \|u\|_{**} \int_0^\infty \frac{r^{n-1}}{(1+r)^n} \Upsilon_{\#}\left(\frac{tr}{\varepsilon}\right) \frac{dr}{r} \\
&\leq C \|u\|_{**} (t/\varepsilon) \int_0^\infty \frac{s^{n-1}}{(t/\varepsilon + s)^n} \Upsilon_{\#}(s) \frac{ds}{s} \\
&= C \|u\|_{**} (t/\varepsilon) \Psi(t/\varepsilon) \\
&\leq C \|u\|_{**} (t/\varepsilon) (1 + \log^+(\varepsilon/t)). \tag{4.30}
\end{aligned}$$

This suits our current purposes.

Consider next the task of estimating  $\nabla v_\varepsilon$ . Using the properties of the Poisson kernel, Theorem 2.3 (recall (2.37) and (2.39)) and Lemmas 4.3, 4.4, 4.5, we write

$$\begin{aligned}
|\nabla v_\varepsilon(x', t)| &= |\nabla(P_t^L * (f_\varepsilon(\cdot) - f_\varepsilon(x')))(x')| \\
&\leq \int_{\mathbb{R}^{n-1}} |\nabla K^L(x' - y', t)| |f_\varepsilon(y') - f_\varepsilon(x')| dy' \\
&\leq C \|u\|_{**} \int_{\mathbb{R}^{n-1}} \frac{1}{(t + |x' - y'|)^n} \Upsilon_{\#}\left(\frac{|x' - y'|}{\varepsilon}\right) dy' \\
&\leq C \|u\|_{**} \varepsilon^{-1} \int_0^\infty \frac{s^{n-1}}{(s + t/\varepsilon)^n} \Upsilon_{\#}(s) \frac{ds}{s} \\
&= C \|u\|_{**} \varepsilon^{-1} \Psi(t/\varepsilon) \\
&\leq C \|u\|_{**} \varepsilon^{-1} (1 + \log^+(\varepsilon/t)). \tag{4.31}
\end{aligned}$$

Collectively, (4.30) and (4.31) prove (4.24).  $\square$

We are now ready to prove that each vertical shift of a smooth null-solution of  $L$  which satisfies a Carleson measure condition in the upper-half space has a Poisson integral representation formula.

**Lemma 4.7.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2) and consider  $P^L$ , the associated Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.3. Let  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$  satisfy  $Lu = 0$  in  $\mathbb{R}_+^n$  and  $\|u\|_{**} < \infty$ . For each given  $\varepsilon > 0$ , define  $u_\varepsilon(x', t) := u(x', t + \varepsilon)$  for every  $(x', t) \in \mathbb{R}_+^n$ .*

*Then for every  $\varepsilon > 0$  one has  $u_\varepsilon \in \mathcal{C}^\infty(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)$ , the restriction  $u_\varepsilon|_{\partial\mathbb{R}_+^n}$  belongs to the space  $L^1\left(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n} dx'\right)^M$ , and the following Poisson integral representation formula holds:*

$$u_\varepsilon(x', t) = \left(P_t^L * (u_\varepsilon|_{\partial\mathbb{R}_+^n})\right)(x'), \quad \forall (x', t) \in \mathbb{R}_+^n. \quad (4.32)$$

*Proof.* For each  $\varepsilon > 0$  set  $f_\varepsilon := u_\varepsilon|_{\partial\mathbb{R}_+^n}$  and note that by part (d) in Lemma 4.6 we have that  $f_\varepsilon$  belongs to  $L^1\left(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n} dx'\right)^M \cap \mathcal{C}^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . Next, for each  $\varepsilon > 0$  define  $v_\varepsilon(x', t) := (P_t^L * f_\varepsilon)(x')$  for every  $(x', t) \in \mathbb{R}_+^n$ . The goal is to show that  $w_\varepsilon := v_\varepsilon - u_\varepsilon \equiv 0$  in  $\mathbb{R}_+^n$ . A key ingredient in this regard is Proposition 2.5.

Notice first that  $w_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$  and  $Lw_\varepsilon = 0$  in  $\mathbb{R}_+^n$  by parts (a) and (e) in Lemma 4.6. Next, we propose to show that  $\text{Tr } w_\varepsilon = 0$ , where  $\text{Tr}$  is as introduced in (2.47). Since by part (a) in Lemma 4.6 we have  $\text{Tr } u_\varepsilon = f_\varepsilon$ , there remains to prove  $\text{Tr } v_\varepsilon = f_\varepsilon$  in  $\mathbb{R}^{n-1}$ . To this end, given  $x' \in \mathbb{R}^{n-1}$ , we use part (f) in Lemma 4.6, the fact that  $f_\varepsilon(x') = u(x', \varepsilon)$ , Lemma 4.3 and Lemma 4.4 (recall that  $\Upsilon_\#$  is defined in (1.44)) to write

$$\begin{aligned} \left| \int_{B((x', 0), r) \cap \mathbb{R}_+^n} v_\varepsilon d\mathcal{L}^n - f_\varepsilon(x') \right| &\leq \int_{B((x', 0), r) \cap \mathbb{R}_+^n} |v_\varepsilon(y', t) - f_\varepsilon(x')| dy' dt \\ &\leq \int_{B((x', 0), r) \cap \mathbb{R}_+^n} |v_\varepsilon(y', t) - f_\varepsilon(y')| dy' dt + \int_{B((x', 0), r) \cap \mathbb{R}_+^n} |f_\varepsilon(y') - f_\varepsilon(x')| dy' dt \\ &\leq C \|u\|_{**} \int_{B((x', 0), r) \cap \mathbb{R}_+^n} (t/\varepsilon) (1 + \log^+(\varepsilon/t)) dy' dt \\ &\quad + C \|u\|_{**} \int_{B((x', 0), r) \cap \mathbb{R}_+^n} \Upsilon_\#(|x' - y'|/\varepsilon) dy' dt \\ &\leq C \|u\|_{**} \frac{r}{\varepsilon} (1 + \log^+(\varepsilon/r)) + C \|u\|_{**} \Upsilon_\#(r/\varepsilon) \longrightarrow 0, \quad \text{as } r \rightarrow 0^+. \end{aligned} \quad (4.33)$$

Thus we conclude that  $\text{Tr } v_\varepsilon(x') = f_\varepsilon(x')$  for every  $x' \in \mathbb{R}^{n-1}$  as desired.

Next we claim that  $w_\varepsilon \in W_{\text{bd}}^{1,2}(\mathbb{R}_+^n, \mathbb{C}^M)$  (recall the latter space from (2.46)). By parts (a) and (c) in Lemma 4.6 we have that  $u_\varepsilon \in W_{\text{bd}}^{1,2}(\mathbb{R}_+^n, \mathbb{C}^M)$ . For  $v_\varepsilon$ , fix  $R > 0$  arbitrary and rely on (4.24) to estimate

$$\begin{aligned} \|v_\varepsilon\|_{L^2(B(0, R) \cap \mathbb{R}_+^n)} &\leq \left( \int_0^R \int_{|x'| \leq R} |v_\varepsilon(x', t) - f_\varepsilon(x')|^2 dx' dt \right)^{\frac{1}{2}} \\ &\quad + \left( \int_0^R \int_{|x'| \leq R} |f_\varepsilon(x')|^2 dx' dt \right)^{\frac{1}{2}} \\ &\leq C \|u\|_{**} (R/\varepsilon) (1 + \log^+(\varepsilon/R)) R^{\frac{n}{2}} \\ &\quad + R^{\frac{1}{2}} \|f_\varepsilon\|_{L^2(B_{n-1}(0', R))} < \infty, \end{aligned} \quad (4.34)$$

since  $f_\varepsilon \in C^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . Above and elsewhere in the paper we make the convention that

$$B_{n-1}(x', R) \text{ denotes the ball in } \mathbb{R}^{n-1} \text{ centered at } x' \in \mathbb{R}^{n-1} \text{ and of radius } R. \quad (4.35)$$

As regards  $\nabla v_\varepsilon$ , we use (4.24) to write

$$\begin{aligned} \|\nabla v_\varepsilon\|_{L^2(B(0,R)\cap\mathbb{R}_+^n)} &\leq C R^{\frac{n-1}{2}} \|u\|_{**} \varepsilon^{-1} \left( \int_0^R (1 + \log^+(\varepsilon/t))^2 dt \right)^{\frac{1}{2}} \\ &= C R^{\frac{n-1}{2}} \|u\|_{**} \varepsilon^{-1} \left( \int_0^\varepsilon (1 + \ln(\varepsilon/t))^2 dt + \int_\varepsilon^R dt \right)^{\frac{1}{2}} \\ &\leq C R^{\frac{n-1}{2}} \|u\|_{**} \varepsilon^{-1} \left( \varepsilon \int_0^1 (1 + \ln(1/s))^2 ds + R \right)^{\frac{1}{2}} \\ &\leq C R^{\frac{n-1}{2}} \|u\|_{**} \varepsilon^{-1} (\varepsilon + R)^{\frac{1}{2}} < \infty. \end{aligned} \quad (4.36)$$

From (4.34) and (4.36) we conclude that  $v_\varepsilon$  and, therefore,  $w_\varepsilon$  belongs to  $W_{\text{bd}}^{1,2}(\mathbb{R}_+^n, \mathbb{C}^M)$ .

Having established these, we may apply Proposition 2.5 and obtain that for every  $z \in \overline{\mathbb{R}_+^n}$  and  $\rho > 0$

$$\begin{aligned} \sup_{\mathbb{R}_+^n \cap B(z, \rho)} |\nabla w_\varepsilon| &\leq C \rho^{-1} \sup_{\mathbb{R}_+^n \cap B(z, 2\rho)} |w_\varepsilon| = C \rho^{-1} \sup_{\mathbb{R}_+^n \cap B(z, 2\rho)} |u_\varepsilon - v_\varepsilon| \\ &\leq C \rho^{-1} \sup_{(y', t) \in \mathbb{R}_+^n \cap B(z, 2\rho)} |u_\varepsilon(y', t) - f_\varepsilon(y')| \\ &\quad + C \rho^{-1} \sup_{(y', t) \in \mathbb{R}_+^n \cap B(z, 2\rho)} |v_\varepsilon(y', t) - f_\varepsilon(y')|. \end{aligned} \quad (4.37)$$

Let  $(y', t) \in \mathbb{R}_+^n \cap B(z, 2\rho)$  and note that Lemma 4.3 implies

$$\begin{aligned} |u_\varepsilon(y', t) - f_\varepsilon(y')| &= |u(y', t + \varepsilon) - u(y', \varepsilon)| \leq \int_\varepsilon^{t+\varepsilon} |\partial_n u(y', \lambda)| d\lambda \\ &\leq C \|u\|_{**} \int_\varepsilon^{t+\varepsilon} \frac{1}{\lambda} d\lambda = C \|u\|_{**} \ln \frac{t + \varepsilon}{\varepsilon}. \end{aligned} \quad (4.38)$$

Proceeding as in (4.30) (the computation leading to the equality in (4.30)), Lemma 4.5 implies that for every  $t > \varepsilon$  we have

$$|v_\varepsilon(y', t) - f_\varepsilon(y')| \leq C \|u\|_{**} (t/\varepsilon) \Psi(t/\varepsilon) \leq C \|u\|_{**} (1 + \ln(t/\varepsilon)). \quad (4.39)$$

Returning with (4.38), (4.39) and (4.24) back to (4.37) we obtain that for every  $z \in \partial\mathbb{R}_+^n$  and every  $\rho > \varepsilon$

$$\begin{aligned} \sup_{\mathbb{R}_+^n \cap B(z, \rho)} |\nabla w_\varepsilon| &\leq C \|u\|_{**} \left( \rho^{-1} \sup_{0 < t < 2\rho} \ln \frac{t + \varepsilon}{\varepsilon} + \rho^{-1} \sup_{0 < t < \varepsilon} (t/\varepsilon) (1 + \log^+(\varepsilon/t)) \right. \\ &\quad \left. + \rho^{-1} \sup_{\varepsilon < t < 2\rho} (1 + \ln(t/\varepsilon)) \right) \\ &\leq C \|u\|_{**} \left( \rho^{-1} \ln \frac{2\rho + \varepsilon}{\varepsilon} + \rho^{-1} + \rho^{-1} (1 + \ln(2\rho/\varepsilon)) \right). \end{aligned} \quad (4.40)$$

Since the last expression converges to 0 as  $\rho \rightarrow \infty$  we obtain that  $\nabla w_\varepsilon \equiv 0$  in  $\mathbb{R}_+^n$ . As we have already shown that  $w_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$  this forces  $w_\varepsilon$  to be constant in  $\mathbb{R}_+^n$ . In concert with the fact that  $\text{Tr } w_\varepsilon = 0$  this ultimately implies  $w_\varepsilon \equiv 0$  in  $\mathbb{R}_+^n$  as desired.  $\square$

Moving on, in Lemmas 4.8-4.11 below we develop tools which are essential in the proof of Proposition 4.12, where we prove a partial converse to part (e) in Proposition 3.1. Concretely, there we show that if  $f \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n} dx'\right)^M$  has the property that the Littlewood-Paley measure  $|\nabla u(x', t)|^2 t dx' dt$  associated with the function  $u$  defined as in (3.3) is a Carleson measure in  $\mathbb{R}_+^n$  then necessarily  $f$  belongs to  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ .

We begin by introducing some notation. Specifically, consider

$$H_a^1(\mathbb{R}^{n-1}) := \left\{ g \in L_c^\infty(\mathbb{R}^{n-1}) : \int_{\mathbb{R}^{n-1}} g d\mathcal{L}^{n-1} = 0 \right\}. \quad (4.41)$$

where  $L_c^\infty(\mathbb{R}^{n-1})$  stands for the space of essentially bounded functions with compact support in  $\mathbb{R}^{n-1}$ . In particular, since any  $g \in H_a^1(\mathbb{R}^{n-1})$  is a scalar multiple of a  $(1, \infty)$ -atom (recall (3.37)), it follows that

$$H_a^1(\mathbb{R}^{n-1}) \text{ is a dense subspace of } H^1(\mathbb{R}^{n-1}). \quad (4.42)$$

In the lemma below we prove a pointwise decay estimate for the vertical maximal operator acting on functions from  $H_a^1(\mathbb{R}^{n-1})$ . Recall the definition from (1.6).

**Lemma 4.8.** *Let  $\phi = (\phi_{\alpha\beta})_{1 \leq \alpha, \beta \leq M} : \mathbb{R}^{n-1} \rightarrow \mathbb{C}^{M \times M}$  be a matrix-valued function with differentiable entries satisfying the property that there exists  $C \in (0, \infty)$  such that*

$$|\phi(x')| + |\nabla \phi(x')| \leq \frac{C}{1 + |x'|^n}, \quad \text{for every } x' \in \mathbb{R}^{n-1}. \quad (4.43)$$

*Pick a function  $g = (g_\alpha)_{1 \leq \alpha \leq M}$  with components in  $H_a^1(\mathbb{R}^{n-1})$ . Then there exists a constant  $C_g \in (0, \infty)$ , depending on  $g$ , such that*

$$\sup_{t>0} |(\phi_t * g)(x')| \leq \frac{C_g}{1 + |x'|^n} \quad \text{for every } x' \in \mathbb{R}^{n-1}. \quad (4.44)$$

*Proof.* Take  $R = R_g \geq 1$  sufficiently large so that (recall (4.35))  $\text{supp } g \subset B_{n-1}(0', R) =: B$ . In the case when  $x' \in 2B$ , for each  $t > 0$  we have

$$\begin{aligned} |(\phi_t * g)(x')| &\leq \|g\|_{L^\infty(\mathbb{R}^{n-1})} \|\phi_t\|_{L^1(\mathbb{R}^{n-1})} = \|g\|_{L^\infty(\mathbb{R}^{n-1})} \|\phi\|_{L^1(\mathbb{R}^{n-1})} \\ &\leq \frac{\|g\|_{L^\infty(\mathbb{R}^{n-1})} \|\phi\|_{L^1(\mathbb{R}^{n-1})} (1 + (2R)^n)}{1 + |x'|^n}. \end{aligned} \quad (4.45)$$

Corresponding to  $x' \notin 2B$ , first we use that  $g$  has vanishing integral and its support condition to write

$$\begin{aligned} |(\phi_t * g)(x')| &\leq \int_{\mathbb{R}^{n-1}} |\phi_t(x' - y') - \phi_t(x')| |g(y')| dy' \\ &\leq \int_B |\phi_t(x' - y') - \phi_t(x')| |g(y')| dy'. \end{aligned} \quad (4.46)$$

Next, we estimate the integrand in the right hand-side. By recalling (1.6), an application of the Mean Value Theorem combined with (4.43), for each  $x' \notin 2B$ ,  $y' \in B$ , and  $t > 0$ , allows us to write

$$\begin{aligned} |\phi_t(x' - y') - \phi_t(x')| &\leq t^{1-n} \frac{|y'|}{t} \sup_{\theta \in [0,1]} |\nabla \phi((x' - \theta y')/t)| \\ &\leq C |y'| \sup_{\theta \in [0,1]} \frac{1}{|x' - \theta y'|^n}. \end{aligned} \quad (4.47)$$



Moreover, whenever  $x' \notin 2B$  and  $y' \in B$ , for each  $\theta \in [0, 1]$  we have

$$|x'| \leq |x' - \theta y'| + \theta |y'| \leq |x' - \theta y'| + R \leq |x' - \theta y'| + \frac{1}{2} |x'|, \quad (4.48)$$

which implies  $|x' - \theta y'| \geq (1/2)|x'| \geq (1/3)(1 + |x'|)$ . The latter when used in (4.47) in combination with (4.46) implies

$$|(\phi * g)(x')| \leq R \frac{C \|g\|_{L^1(\mathbb{R}^{n-1})}}{1 + |x'|^n}, \quad \forall x' \in \mathbb{R}^{n-1} \setminus (2B), \quad \forall t > 0. \quad (4.49)$$

Now the desired conclusion follows from (4.45) and (4.49) by taking

$$C_g := \max \{ \|g\|_{L^\infty(\mathbb{R}^{n-1})} \|\phi\|_{L^1(\mathbb{R}^{n-1})} (1 + (2R)^n), CR \|g\|_{L^1(\mathbb{R}^{n-1})} \}. \quad (4.50)$$

The proof of the lemma is therefore complete.  $\square$

Our next preparatory lemma is needed in the proof of Proposition 4.12.

**Lemma 4.9.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2) and consider  $P^L$ , the associated Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.3, as well as  $K^L$ , defined in (2.37). Write*

$$\Phi(x') := (\partial_n K^L)(x', 1) \quad \text{for every } x' \in \mathbb{R}^{n-1}, \quad (4.51)$$

and, whenever  $0 < a < b < \infty$ , also set

$$\Psi_{a,b}(x') := 4 \int_a^b (\Phi_t * \Phi_t)(x') \frac{dt}{t}, \quad \forall x' \in \mathbb{R}^{n-1}. \quad (4.52)$$

Then, whenever  $0 < a < b < \infty$  there holds

$$\Psi_{a,b}(x') = \Phi_{2b}(x') - P_{2b}^L(x') - \Phi_{2a}(x') + P_{2a}^L(x'), \quad \forall x' \in \mathbb{R}^{n-1}. \quad (4.53)$$

*Proof.* Since  $\nabla K^L$  is homogeneous of order  $-n$  (recall item (5) in Theorem 2.3), for every  $(x', t) \in \mathbb{R}_+^n$  we may write

$$\Phi_t(x') = t^{1-n} \Phi(x'/t) = t^{1-n} (\partial_n K^L)(x'/t, 1) = t (\partial_n K^L)(x', t) = t \partial_t K^L(x', t). \quad (4.54)$$

Consequently, in view of definition (2.37), in the current notation we have

$$\Phi_t(x') = t \partial_t [P_t^L(x')], \quad \forall (x', t) \in \mathbb{R}_+^n. \quad (4.55)$$

Fix  $h \in \mathcal{C}_0^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . Observe that

$$(\Psi_{a,b} * h)(x') = 4 \int_a^b \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \Phi_t(x' - z' - y') \Phi_t(z') h(y') dz' dy' \frac{dt}{t} \quad (4.56)$$

since the triple integral is absolutely convergent in view of the assumptions made on  $h$  and (2.39). Set  $u(x', t) := (P_t^L * h)(x')$  for each  $(x', t) \in \mathbb{R}_+^n$  and in light of (4.55) further write (4.56) in the form

$$\begin{aligned} (\Psi_{a,b} * h)(x') &= 4 \int_a^b \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \partial_n K^L(x' - z' - y', t) \partial_n K^L(z', t) h(y') dz' dy' t dt \\ &= 4 \int_a^b \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \partial_n K^L(z', t) \partial_n K^L(x' - z' - y', t) h(y') dy' dz' t dt \\ &= 4 \int_a^b \int_{\mathbb{R}^{n-1}} \partial_n K^L(z', t) \partial_n u(x' - z', t) dz' t dt. \end{aligned} \quad (4.57)$$

Next, for every  $(x', t) \in \mathbb{R}_+^n$ , define  $v(x', t) := (\partial_n u)(x', t)$ . By part (7) in Theorem 2.3 we have that  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$  and  $Lu = 0$  in  $\mathbb{R}_+^n$ . In turn, these imply  $v \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$  and  $Lv = 0$  in  $\mathbb{R}_+^n$ .

Moving on, for each  $s > 0$  set  $v_s(x', t) := v(x', t + s)$  for every  $(x', t) \in \mathbb{R}_+^n$ . Then we have  $v_s \in \mathcal{C}^\infty(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)$  and  $Lv_s = 0$  in  $\mathbb{R}_+^n$ . Now recall (1.64). For  $\kappa \in (0, \infty)$  arbitrary, if  $x' \in \mathbb{R}^{n-1}$  is fixed, Theorem 2.4 allows us to estimate

$$\begin{aligned} |v_s(y', t)| &= |(\partial_n u)(y', t + s)| \leq \frac{C}{s} \int_{B((y', t+s), \kappa s / \sqrt{1+\kappa^2})} |u| d\mathcal{L}^n \\ &\leq \frac{C}{s} \mathcal{N}u(x'), \quad \forall (y', t) \in \Gamma_\kappa(x'), \end{aligned} \quad (4.58)$$

where for the last inequality we have used that  $B((y', t + s), \kappa s / \sqrt{1 + \kappa^2}) \subset \Gamma_\kappa(x')$ . Hence, (4.58) combined with (2.40) yields

$$(\mathcal{N}v_s)(x') \leq \frac{C}{s} (\mathcal{N}u)(x') \leq \frac{C}{s} (\mathcal{M}h)(x'), \quad \forall x' \in \mathbb{R}^{n-1}. \quad (4.59)$$

Upon recalling that  $h \in \mathcal{C}_0^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M)$  and that the Hardy-Littlewood maximal operator is bounded on  $L^p(\mathbb{R}^{n-1})$  for  $p \in (1, \infty)$ , from (4.59) we may infer that  $\mathcal{N}v_s \in L^p(\mathbb{R}^{n-1})$  for every  $p \in (1, \infty)$ . In view of all these, we may apply Corollary 2.6 to  $v_s$  and obtain that for each  $s \in (0, \infty)$

$$\begin{aligned} v_s(x', t) &= (P_t^L * (v_s|_{\partial\mathbb{R}_+^n}))(x') = \int_{\mathbb{R}^{n-1}} P_t^L(z') v_s(x' - z', 0) dz' \\ &= \int_{\mathbb{R}^{n-1}} K^L(z', t) v_s(x' - z', 0) dz' \\ &= \int_{\mathbb{R}^{n-1}} K^L(z', t) (\partial_n u)(x' - z', s) dz', \quad \forall (x', t) \in \mathbb{R}_+^n. \end{aligned} \quad (4.60)$$

Thus, for every  $(x', t) \in \mathbb{R}_+^n$  and every  $s > 0$  we have

$$(\partial_n^2 u)(x', t + s) = \partial_n v_s(x', t) = \int_{\mathbb{R}^{n-1}} \partial_n K^L(z', t) (\partial_n u)(x' - z', s) dz'. \quad (4.61)$$

Applying (4.61) with  $s = t$  and substituting the resulting equality into (4.57) and making use of (4.55) we obtain

$$\begin{aligned} (\Psi_{a,b} * h)(x') &= 4 \int_a^b (\partial_n^2 u)(x', 2t) t dt = 4 \left[ \frac{t}{2} (\partial_n u)(x', 2t) - \frac{1}{4} u(x', 2t) \right]_{t=a}^{t=b} \\ &= \left[ \Phi_{2t} * h(x') - P_{2t}^L * h(x') \right]_{t=a}^{t=b}. \end{aligned} \quad (4.62)$$

This readily yields

$$(\Psi_{a,b} * h)(x') = (\Phi_{2b} * h)(x') - (P_{2b}^L * h)(x') - (\Phi_{2a} * h)(x') + (P_{2a}^L * h)(x') \quad (4.63)$$

for every  $x' \in \mathbb{R}^{n-1}$ . Note that (4.63) holds for every  $h \in \mathcal{C}_0^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M)$  and therefore (4.53) holds for a.e.  $x' \in \mathbb{R}^{n-1}$ . In addition, by Theorem 2.3 and the fact that  $0 < a < b < \infty$  we see that both sides of (4.53) are continuous functions in  $\mathbb{R}^{n-1}$ . Consequently, the desired equality holds everywhere. The proof of the lemma is complete.  $\square$

Given a Lebesgue measurable function  $F : \mathbb{R}_+^n \rightarrow \mathbb{C}$ , for every  $x' \in \mathbb{R}^{n-1}$  introduce the Lusin area-function

$$(\mathcal{A}F)(x') := \left( \int_{\Gamma_\kappa(x')} |F(y', t)|^2 \frac{dy' dt}{t^n} \right)^{\frac{1}{2}} \quad (4.64)$$

and the Carleson operator

$$(\mathcal{C}F)(x') := \sup_{Q \ni x'} \left( \int_0^{\ell(Q)} \int_Q |F(y', t)|^2 \frac{dy' dt}{t} \right)^{\frac{1}{2}}. \quad (4.65)$$

In relation to these operators, in Lemma 4.10 below we recall the result from [8, Theorem 1, p. 313]. The statement in [8] contains as assumptions the requirements  $\mathcal{C}F \in L^\infty(\mathbb{R}^{n-1})$  and  $\mathcal{A}G \in L^1(\mathbb{R}^{n-1})$ . However, these said assumptions may be removed by a suitable truncation and limiting argument. As a matter of fact, in the setting in which we apply Lemma 4.10 the conditions  $\mathcal{C}F \in L^\infty(\mathbb{R}^{n-1})$  and  $\mathcal{A}G \in L^1(\mathbb{R}^{n-1})$  are automatically satisfied.

**Lemma 4.10.** *There exists some constant  $C \in (0, \infty)$  such that for any Lebesgue measurable functions  $F, G : \mathbb{R}_+^n \rightarrow \mathbb{C}$  there holds*

$$\int_{\mathbb{R}_+^n} |F(x', t) G(x', t)| \frac{dx' dt}{t} \leq C \int_{\mathbb{R}^{n-1}} \mathcal{C}F(x') \mathcal{A}G(x') dx'. \quad (4.66)$$

To state the final preparatory lemma required in the proof of Proposition 4.12, one more piece of notation is needed. In the sequel,  $A^\top$  denotes the transpose of a given matrix  $A$ .

**Lemma 4.11.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2) and consider  $P^L$ , the associated Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.3, as well as  $K^L$  as in (2.37). Recall  $\Phi$  from (4.51) and for each  $x' \in \mathbb{R}^{n-1}$  set  $\tilde{\Phi}(x') := \Phi^\top(-x')$ . Furthermore fix  $\kappa \in (0, \infty)$  arbitrary and, given a function  $f = (f_\beta)_{1 \leq \beta \leq M} : \mathbb{R}^{n-1} \rightarrow \mathbb{C}^M$  with Lebesgue measurable entries, define for each  $x' \in \mathbb{R}^{n-1}$*

$$\begin{aligned} (S_{\tilde{\Phi}} f)(x') &:= \left( \int_{\Gamma_\kappa(x')} |(\tilde{\Phi}_t * f)(y')|^2 \frac{dy' dt}{t^n} \right)^{\frac{1}{2}} \\ &= \left( \left( \int_{\Gamma_\kappa(x')} \sum_{\beta=1}^M |((\tilde{\Phi}_t)_{\alpha\beta} * f_\beta)(y')|^2 \frac{dy' dt}{t^n} \right)^{\frac{1}{2}} \right)_{1 \leq \alpha \leq M}. \end{aligned} \quad (4.67)$$

Then  $S_{\tilde{\Phi}}$  is a bounded operator from  $H^1(\mathbb{R}^{n-1}, \mathbb{C}^M)$  into  $L^1(\mathbb{R}^{n-1})$ .

*Proof.* For each  $\alpha, \beta \in \{1, \dots, M\}$ , write  $\theta_{\alpha\beta}(x', t; y') := t \partial_n K_{\beta\alpha}^L(x' - y', t)$  for every  $x', y' \in \mathbb{R}^{n-1}$  and  $t > 0$ , and denote by  $\Theta_{\alpha\beta}$  the integral operator as in (3.18) corresponding to  $\theta_{\alpha\beta}$  in place of  $\theta$ . Notice that (3.76), (3.77) and (3.78) (with  $j = n$  and the roles of  $\alpha$  and  $\beta$  reversed) allow us to apply Proposition 3.3 and write

$$\begin{aligned} \|S_{\tilde{\Phi}} f\|_{L^1(\mathbb{R}^{n-1})} &\leq \sum_{1 \leq \alpha, \beta \leq M} \|S_{\Theta_{\alpha\beta}} f_\beta\|_{L^1(\mathbb{R}^{n-1})} \\ &\leq C \sum_{1 \leq \beta \leq M} \|f_\beta\|_{H^1(\mathbb{R}^{n-1})} = C \|f\|_{H^1(\mathbb{R}^{n-1}, \mathbb{C}^M)}. \end{aligned} \quad (4.68)$$

The desired conclusion now follows from (4.68).  $\square$

We have seen in Proposition 3.1 part (e) that if  $f \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  then the Littlewood-Paley measure  $|\nabla u(x', t)|^2 t dx' dt$  associated with the function  $u$  defined as in (3.3) is a Carleson measure in  $\mathbb{R}_+^n$  (cf. (1.10)). In the proposition below we shall establish the converse implication along with the estimate which naturally accompanies this statement. In the proof, Lemmas 4.8-4.11 as well as the fundamental duality result from [13] asserting that

$$(H^1(\mathbb{R}^{n-1}, \mathbb{C}^M))^* = \widetilde{\text{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M) \quad (4.69)$$

are going to play a key role.

**Proposition 4.12.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2) and consider  $P^L$ , the associated Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.3, together with  $K^L$  as in (2.37). Recall  $\Phi$  from (4.51). Let  $f \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n} dx'\right)^M$  and consider the measure in  $\mathbb{R}_+^n$  defined by*

$$d\mu(x', t) := |(\Phi_t * f)(x')|^2 \frac{dx' dt}{t}. \quad (4.70)$$

Then whenever  $\mu$  is a Carleson measure, that is,

$$\|\mu\|_{\mathcal{C}(\mathbb{R}_+^n)} = \sup_{Q \subset \mathbb{R}^{n-1}} \int_0^{\ell(Q)} \int_Q |(\Phi_t * f)(x')|^2 \frac{dx' dt}{t} < \infty, \quad (4.71)$$

one necessarily has  $f \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  and

$$\|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)}^2 \leq C \|\mu\|_{\mathcal{C}(\mathbb{R}_+^n)} \quad (4.72)$$

for some constant  $C \in (0, \infty)$  independent of  $f$ .

*Proof.* Fix a function  $f$  as in the hypotheses of the proposition and suppose  $\mu$  satisfies (4.71). Let  $g \in H_a^1(\mathbb{R}^{n-1})$  (see (4.41)) and for some arbitrary  $\alpha_0 \in \{1, \dots, M\}$  define

$$h := (g \delta_{\alpha\alpha_0})_{1 \leq \alpha \leq M} \in [H_a^1(\mathbb{R}^{n-1})]^M \subset H^1(\mathbb{R}^{n-1}, \mathbb{C}^M), \quad (4.73)$$

where  $\delta_{\alpha\alpha_0}$  denotes the standard Kronecker symbol.

Next, recall the expression of the classical harmonic Poisson kernel (that is, the Poisson kernel associated with the Laplacian  $\Delta$ )

$$P^\Delta(x') := \frac{2}{\omega_{n-1}} \frac{1}{(1 + |x'|^2)^{\frac{n}{2}}}, \quad \forall x' \in \mathbb{R}^{n-1}, \quad (4.74)$$

where  $\omega_{n-1}$  stands for the area of the unit sphere in  $\mathbb{R}^n$ . Then the definition of  $\Phi$ , (2.39) in Theorem 2.3, and (4.74) imply

$$|\Phi_t(x')| \leq C P_t^\Delta(x'), \quad \forall x' \in \mathbb{R}^{n-1}, \forall t \in (0, \infty). \quad (4.75)$$

Also, by the semigroup property (cf., e.g., [34, (vi), p. 62], or part (8) in Theorem 2.3), for every  $\varepsilon \in (0, 1)$  and every  $t \in (\varepsilon, \varepsilon^{-1})$  we have

$$P_t^\Delta * P_t^\Delta = P_{2t}^\Delta \leq C_\varepsilon P^\Delta. \quad (4.76)$$

Combining (4.75) and (4.76), for each  $\varepsilon \in (0, 1)$  we may write

$$\begin{aligned} & \int_\varepsilon^{\varepsilon^{-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} |\Phi_t(x' - y' - z')| |\Phi_t(z')| |f(y')| |h(x')| dz' dy' dx' \frac{dt}{t} \\ & \leq C \int_\varepsilon^{\varepsilon^{-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} P_t^\Delta(x' - y' - z') P_t^\Delta(z') |f(y')| |g(x')| dz' dy' dx' \frac{dt}{t} \\ & \leq C_\varepsilon \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} P^\Delta(x' - y') |f(y')| |g(x')| dy' dx' \\ & \leq C_\varepsilon \left( \int_{\mathbb{R}^{n-1}} (1 + |x'|^n) |g(x')| dx' \right) \left( \int_{\mathbb{R}^{n-1}} \frac{|f(y')|}{1 + |y'|^n} dx' \right) < \infty, \end{aligned} \quad (4.77)$$

where for the last inequality we have used the fact that  $1 + |y'| \leq (1 + |x'|)(1 + |x' - y'|)$  for every  $x', y' \in \mathbb{R}^{n-1}$ , while the finiteness of the rightmost term in (4.77) follows from our assumptions on  $f$  and  $g$ . Thus, recalling the definition of  $\Psi_{\varepsilon, \varepsilon^{-1}}$  from (4.52) we have that

$$\int_{\mathbb{R}^{n-1}} \langle (\Psi_{\varepsilon, \varepsilon^{-1}} * f)(x'), h(x') \rangle dx'$$

$$= \int_{\varepsilon}^{\varepsilon^{-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \langle \Phi_t(x' - y' - z') \Phi_t(z') f(y'), h(x') \rangle dz' dy' dx' \frac{dt}{t} \quad (4.78)$$

is an absolutely convergent integral. Here and elsewhere we use the notation

$$\langle \lambda, \lambda' \rangle := \sum_{\alpha=1}^M \lambda_{\alpha} \lambda'_{\alpha}, \quad \lambda = (\lambda_{\alpha})_{1 \leq \alpha \leq M}, \quad \lambda' = (\lambda'_{\alpha})_{1 \leq \alpha \leq M} \in \mathbb{C}^M. \quad (4.79)$$

To continue, we introduce the (matrix valued) functions

$$\tilde{\Phi}(x') := \Phi^{\top}(-x'), \quad \tilde{\Psi}_{\varepsilon, \varepsilon^{-1}}(x') := \Psi_{\varepsilon, \varepsilon^{-1}}^{\top}(-x'), \quad \text{and} \quad \tilde{P}^L(x') := (P^L)^{\top}(-x'), \quad (4.80)$$

defined for every  $x' \in \mathbb{R}^{n-1}$ . Then, for every  $\varepsilon > 0$ , we may write

$$\begin{aligned} \left| \int_{\mathbb{R}^{n-1}} \langle f(x'), (\tilde{\Psi}_{\varepsilon, \varepsilon^{-1}} * h)(x') \rangle dx' \right| &= \left| \int_{\mathbb{R}^{n-1}} \langle (\Psi_{\varepsilon, \varepsilon^{-1}} * f)(x'), h(x') \rangle dx' \right| \\ &= \left| \int_{\varepsilon}^{\varepsilon^{-1}} \int_{\mathbb{R}^{n-1}} \langle (\Phi_t * \Phi_t * f)(x'), h(x') \rangle dx' \frac{dt}{t} \right| \\ &= \left| \int_{\varepsilon}^{\varepsilon^{-1}} \int_{\mathbb{R}^{n-1}} \langle (\Phi_t * f)(x'), (\tilde{\Phi}_t * h)(x') \rangle dx' \frac{dt}{t} \right| \\ &= \left| \int_{\varepsilon}^{\varepsilon^{-1}} \int_{\mathbb{R}^{n-1}} \langle F(x', t), H(x', t) \rangle dx' \frac{dt}{t} \right| \\ &\leq \int_{\mathbb{R}_+^n} |\langle F(x', t), H(x', t) \rangle| dx' \frac{dt}{t} \end{aligned} \quad (4.81)$$

where  $F(x', t) := (\Phi_t * f)(x')$  and  $H(x', t) := (\tilde{\Phi}_t * h)(x')$  for every  $(x', t) \in \mathbb{R}_+^n$ . Denote by  $(F_{\alpha})_{1 \leq \alpha \leq M}$  and  $(H_{\alpha})_{1 \leq \alpha \leq M}$  the scalar components of  $F$  and  $H$ , respectively. Note that (4.65), the definition of  $F$ , and (4.71) imply

$$\|\mathcal{C}F_{\alpha}\|_{L^{\infty}(\mathbb{R}^{n-1})} \leq \|\mu\|_{\mathcal{C}(\mathbb{R}_+^n)}^{\frac{1}{2}} < \infty, \quad \forall \alpha \in \{1, \dots, M\}. \quad (4.82)$$

Also, (4.73), (4.64), and Lemma 4.11 allow us to infer that

$$\begin{aligned} \|\mathcal{A}H_{\alpha}\|_{L^1(\mathbb{R}^{n-1})} &\leq \|S_{\tilde{\Phi}}h\|_{L^1(\mathbb{R}^{n-1})} \leq C\|h\|_{H^1(\mathbb{R}^{n-1}, \mathbb{C}^M)} \\ &= C\|g\|_{H^1(\mathbb{R}^{n-1})}, \quad \forall \alpha \in \{1, \dots, M\}. \end{aligned} \quad (4.83)$$

Consequently, Lemma 4.10, (4.82), and (4.83) allow us to estimate

$$\begin{aligned} \int_{\mathbb{R}_+^n} |\langle F(x', t), H(x', t) \rangle| dx' \frac{dt}{t} &\leq \sum_{\alpha=1}^M \int_{\mathbb{R}_+^n} |F_{\alpha}(x', t) H_{\alpha}(x', t)| dx' \frac{dt}{t} \\ &\leq C \sum_{\alpha=1}^M \int_{\mathbb{R}^{n-1}} \mathcal{C}F_{\alpha}(x') \mathcal{A}H_{\alpha}(x') dx' \\ &\leq C \sum_{\alpha=1}^M \|\mathcal{C}F_{\alpha}\|_{L^{\infty}(\mathbb{R}^{n-1})} \|\mathcal{A}H_{\alpha}\|_{L^1(\mathbb{R}^{n-1})} \\ &= C \|\mu\|_{\mathcal{C}(\mathbb{R}_+^n)}^{\frac{1}{2}} \|g\|_{H^1(\mathbb{R}^{n-1})}. \end{aligned} \quad (4.84)$$

At this point we make the claim that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^{n-1}} \langle f(x'), (\tilde{\Psi}_{\varepsilon, \varepsilon^{-1}} * h)(x') \rangle dx' = \int_{\mathbb{R}^{n-1}} \langle f(x'), h(x') \rangle dx'. \quad (4.85)$$

The idea is to show that Lebesgue's Dominated Convergence Theorem applies in our setting. With this goal in mind, first observe that by part (5) in Theorem 2.3, for every multi-index  $\alpha \in \mathbb{N}_0^{n-1}$ , we have

$$|\partial^\alpha \Phi(x')| = |\partial^\alpha \partial_n K^L(x', 1)| \leq C_\alpha |(x', 1)|^{-n-|\alpha|}, \quad (4.86)$$

hence  $\tilde{\Phi}$  satisfies the hypotheses of Lemma 4.8. Moreover, by parts (1) and (5) in Theorem 2.3 we also have that  $\tilde{P}^L$  satisfies the hypotheses of Lemma 4.8. Hence, Lemma 4.8 and (4.73) give

$$\sup_{t>0} |(\tilde{\Phi}_t * h)(x')| + \sup_{t>0} |(\tilde{P}_t^L * h)(x')| \leq \frac{C_h}{1 + |x'|^n}, \quad \text{for every } x' \in \mathbb{R}^{n-1}. \quad (4.87)$$

In light of (4.53), the latter yields

$$\sup_{0 < \varepsilon < 1} |(\tilde{\Psi}_{\varepsilon, \varepsilon^{-1}} * h)(x')| \leq \frac{C_h}{1 + |x'|^n}, \quad \text{for every } x' \in \mathbb{R}^{n-1}. \quad (4.88)$$

From this and the fact that  $f \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n} dx'\right)^M$  we arrive at the conclusion that

$$\sup_{0 < \varepsilon < 1} |\langle f, \tilde{\Psi}_{\varepsilon, \varepsilon^{-1}} * h \rangle| \in L^1(\mathbb{R}^{n-1}). \quad (4.89)$$

Next, we focus on the pointwise convergence of the functions under the integral in the left hand-side of (4.85). First, by (2.34), (2.55) in Lemma 2.7, and (2.35) in Theorem 2.3 we obtain

$$\lim_{s \rightarrow 0^+} (\tilde{P}_s^L * h)(x') = \left( \int_{\mathbb{R}^{n-1}} \tilde{P}^L(y') dy' \right) h(x') = h(x'), \quad \text{for a.e. } x' \in \mathbb{R}^{n-1}. \quad (4.90)$$

Second, using a suitable change of variables, the properties of  $h$ , and Lebesgue's Dominated Convergence Theorem we have

$$\lim_{s \rightarrow \infty} (\tilde{P}_s^L * h)(x') = \lim_{s \rightarrow \infty} \int_{\mathbb{R}^{n-1}} \tilde{P}^L(y') h(x' - sy') dy' = 0. \quad (4.91)$$

Third, by (3.72) for every  $t > 0$  we have

$$\int_{\mathbb{R}^{n-1}} (\partial_n K)(x', t) dx' = 0, \quad \forall t > 0. \quad (4.92)$$

which when specialized to  $t = 1$  yields

$$\int_{\mathbb{R}^{n-1}} \tilde{\Phi}(x') dx' = \left( \int_{\mathbb{R}^{n-1}} \Phi(-x') dx' \right)^\top = \left( \int_{\mathbb{R}^{n-1}} \Phi(x') dx' \right)^\top = 0. \quad (4.93)$$

This, (4.86), and Lemma 2.7 applied to  $\tilde{\Phi}$  then give that

$$\lim_{s \rightarrow 0^+} (\tilde{\Phi}_s * h)(x') = \left( \int_{\mathbb{R}^{n-1}} \tilde{\Phi}(y') dy' \right) h(x') = 0 \quad \text{for a.e. } x' \in \mathbb{R}^{n-1}. \quad (4.94)$$

Fourth, a suitable change of variables, the properties of  $h$ , and Lebesgue's Dominated Convergence Theorem also yield

$$\lim_{s \rightarrow \infty} (\tilde{\Phi}_s * h)(x') = \lim_{s \rightarrow \infty} \int_{\mathbb{R}^{n-1}} \tilde{\Phi}(y') h(x' - sy') dy' = 0. \quad (4.95)$$

In concert, (4.90), (4.91), (4.94), (4.95), and (4.53) imply the pointwise convergence

$$\lim_{\varepsilon \rightarrow 0^+} (\tilde{\Psi}_{\varepsilon, \varepsilon^{-1}} * h)(x') = h(x') \quad \text{for a.e. } x' \in \mathbb{R}^{n-1}. \quad (4.96)$$

Having dispensed of (4.89) and (4.96), we may apply Lebesgue's Dominated Theorem to write

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^{n-1}} \langle f(x'), (\tilde{\Psi}_{\varepsilon, \varepsilon^{-1}} * h)(x') \rangle dx' &= \int_{\mathbb{R}^{n-1}} \left\langle f(x'), \lim_{\varepsilon \rightarrow 0^+} (\tilde{\Psi}_{\varepsilon, \varepsilon^{-1}} * h)(x') \right\rangle dx' \\ &= \int_{\mathbb{R}^{n-1}} \langle f(x'), h(x') \rangle dx', \end{aligned} \quad (4.97)$$

finishing the proof of the claim in (4.85).

From the definition of  $h$ , (4.85), (4.81), and (4.84) we may conclude that

$$\left| \int_{\mathbb{R}^{n-1}} f_{\alpha_0}(x') g(x') dx' \right| \leq C \|\mu\|_{\mathcal{C}(\mathbb{R}_+^n)}^{\frac{1}{2}} \|g\|_{H^1(\mathbb{R}^{n-1})}. \quad (4.98)$$

In particular, if we define the functional  $\Lambda_f^{\alpha_0} : H_a^1(\mathbb{R}^{n-1}) \rightarrow \mathbb{C}$  by setting

$$\Lambda_f^{\alpha_0}(g) := \int_{\mathbb{R}^{n-1}} f_{\alpha_0} g d\mathcal{L}^{n-1} \quad \text{for every } g \in H_a^1(\mathbb{R}^{n-1}), \quad (4.99)$$

then (4.98) implies  $\Lambda_f^{\alpha_0} \in (H^1(\mathbb{R}^{n-1}))^*$  (here we also used (4.42)). Recalling (4.69), it follows that

$$\begin{aligned} &\text{there exists } b_{\alpha_0} \in \text{BMO}(\mathbb{R}^{n-1}) \text{ such that } \|b_{\alpha_0}\|_{\text{BMO}(\mathbb{R}^{n-1})} \leq C \|\mu\|_{\mathcal{C}(\mathbb{R}_+^n)}^{\frac{1}{2}} \\ &\text{and } \Lambda_f^{\alpha_0}(g) = \int_{\mathbb{R}^{n-1}} b_{\alpha_0} g d\mathcal{L}^{n-1} \quad \text{for every function } g \in H_a^1(\mathbb{R}^{n-1}). \end{aligned} \quad (4.100)$$

Thus,

$$\int_{\mathbb{R}^{n-1}} (b_{\alpha_0} - f_{\alpha_0}) g d\mathcal{L}^{n-1} = 0, \quad \forall g \in H_a^1(\mathbb{R}^{n-1}). \quad (4.101)$$

Hence, if we set  $v_{\alpha_0} := b_{\alpha_0} - f_{\alpha_0}$ , then (4.101) implies that

$$v_{\alpha_0} \text{ is constant almost everywhere in } \mathbb{R}^{n-1}. \quad (4.102)$$

Indeed, if the latter were not true, one could find two bounded Lebesgue measurable sets  $E^+$ ,  $E^-$  in  $\mathbb{R}^{n-1}$  such that  $0 < |E^\pm| < \infty$  and  $v_{\alpha_0}(x') \leq a < b \leq v_{\alpha_0}(y')$  for all  $x' \in E^-$ ,  $y' \in E^+$ . Then the function

$$g := \frac{\mathbf{1}_{E^+}}{|E^+|} - \frac{\mathbf{1}_{E^-}}{|E^-|} \quad \text{belongs to } H_a^1(\mathbb{R}^{n-1}) \quad (4.103)$$

and, when used in (4.101), forces

$$0 = \int_{\mathbb{R}^{n-1}} v_{\alpha_0} g d\mathcal{L}^{n-1} \geq b - a > 0. \quad (4.104)$$

This contradiction proves (4.102). In summary, we have shown that  $b_{\alpha_0} - f_{\alpha_0}$  is constant almost everywhere in  $\mathbb{R}^{n-1}$ . Upon recalling the first line in (4.100), it follows that  $f_{\alpha_0} \in \text{BMO}(\mathbb{R}^{n-1})$  with

$$\|f_{\alpha_0}\|_{\text{BMO}(\mathbb{R}^{n-1})} \leq C \|\mu\|_{\mathcal{C}(\mathbb{R}_+^n)}^{\frac{1}{2}}. \quad (4.105)$$

Since  $\alpha_0 \in \{1, \dots, M\}$  is arbitrary, we may further conclude that  $f \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  and satisfies (4.72), as wanted.  $\square$

In turn, Proposition 4.12 is one of the main ingredients in the proof of the fact that the boundary traces of vertical shifts of a smooth null-solution of  $L$  satisfying a Carleson measure condition in the upper-half space belong to BMO, uniformly with respect to the shift.



**Lemma 4.13.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2) and consider  $P^L$ , the associated Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.3. Suppose  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$  satisfies  $Lu = 0$  in  $\mathbb{R}_+^n$  and  $\|u\|_{**} < \infty$ . For each  $\varepsilon > 0$ , set  $u_\varepsilon(x', t) := u(x', t + \varepsilon)$ , for every  $(x', t) \in \mathbb{R}_+^n$  and  $f_\varepsilon := u_\varepsilon|_{\partial\mathbb{R}_+^n}$ . Then for each  $\varepsilon > 0$  we have  $f_\varepsilon \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  and*

$$\|f_\varepsilon\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq C \|u\|_{**}, \quad (4.106)$$

for some  $C \in (0, \infty)$  independent of  $\varepsilon$ .

*Proof.* We are going to apply Proposition 4.12 to  $f_\varepsilon$ . Note first that by part (d) in Lemma 4.6 we have  $f_\varepsilon \in L^1\left(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n} dx'\right)^M \cap \mathcal{C}^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . Hence we may define  $\mu_\varepsilon$  as in (4.70) associated with  $f_\varepsilon$ , where we recall that  $\Phi(x') = \partial_n K^L(x', 1)$  for every  $x' \in \mathbb{R}^{n-1}$  and  $K^L(x', t) = t^{1-n} P^L(x'/t)$  for every  $x' \in \mathbb{R}_+^n$ . Also, Lemma 4.7 and (4.55) imply

$$t \partial_t u_\varepsilon(x', t) = t \partial_t (P_t^L * f_\varepsilon)(x') = (\Phi_t * f_\varepsilon)(x'), \quad \forall (x', t) \in \mathbb{R}_+^n. \quad (4.107)$$

Thus part (b) in Lemma 4.6 yields

$$\begin{aligned} \|\mu_\varepsilon\|_{\mathcal{C}(\mathbb{R}_+^n)} &= \sup_{Q \subset \mathbb{R}^{n-1}} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\Phi_t * f_\varepsilon(x')|^2 \frac{dx' dt}{t} \\ &= \sup_{Q \subset \mathbb{R}^{n-1}} \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\partial_t u_\varepsilon(x', t)|^2 t dx' dt \\ &\leq \|u_\varepsilon\|_{**}^2 \leq C \|u\|_{**}^2 < \infty. \end{aligned} \quad (4.108)$$

Consequently, we may invoke Proposition 4.12 to conclude that  $f_\varepsilon \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  and

$$\|f_\varepsilon\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq C \|\mu_\varepsilon\|_{\mathcal{C}(\mathbb{R}_+^n)}^{\frac{1}{2}} \leq C \|u\|_{**}, \quad (4.109)$$

as wanted.  $\square$

The aim in Lemma 4.14 below is to show that derivatives of the kernel function  $K^L$  are molecules in the sense of Hardy space theory. To make this precise, let us recall the definition of  $L^2(\mathbb{R}^{n-1})$ -molecules for the Hardy space  $H^1(\mathbb{R}^{n-1})$ . Specifically, given  $\varepsilon > 0$  and a ball  $B \subset \mathbb{R}^{n-1}$ , a function  $m \in L^1(\mathbb{R}^{n-1})$  is said to be an  $(L^2(\mathbb{R}^{n-1}), \varepsilon)$ -molecule relative to  $B$  provided

$$\int_{\mathbb{R}^{n-1}} m(x') dx' = 0 \quad (4.110)$$

and

$$\|m\|_{L^2(B)} \leq |B|^{-\frac{1}{2}}, \quad \|m\|_{L^2(2^k B \setminus 2^{k-1} B)} \leq |2^k B|^{-\frac{1}{2}} 2^{-k\varepsilon}, \quad \forall k \in \mathbb{N}. \quad (4.111)$$

**Lemma 4.14.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2) and consider  $P^L$ , the associated Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.3. Then there exists a constant  $C \in (0, \infty)$  such that for any fixed  $t > 0$ , the components of  $C t \nabla K^L(\cdot, t)$  are  $(L^2(\mathbb{R}^{n-1}), 1)$ -molecules relative to  $B_{n-1}(0', t)$ . In particular,*

$$\sup_{t>0} \|t \nabla K^L(\cdot, t)\|_{H^1(\mathbb{R}^{n-1})} < \infty. \quad (4.112)$$

Consequently, if  $f \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  and the sequence  $\{f_k\}_{k \in \mathbb{N}} \subset \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  is such that  $[f_k] \rightarrow [f]$  in the weak- $*$  topology on  $\widetilde{\text{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  as  $k \rightarrow \infty$ , i.e.,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^{n-1}} f_k g d\mathcal{L}^{n-1} = \int_{\mathbb{R}^{n-1}} f g d\mathcal{L}^{n-1} \quad \forall g \in H^1(\mathbb{R}^{n-1}, \mathbb{C}^M), \quad (4.113)$$

then for every  $(x', t) \in \mathbb{R}_+^n$  fixed one has

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^{n-1}} t \nabla K^L(x' - y', t) f_k(y') dy' = \int_{\mathbb{R}^{n-1}} t \nabla K^L(x' - y', t) f(y') dy'. \quad (4.114)$$

*Proof.* Fix  $t > 0$ , set  $B_t := B_{n-1}(0', t)$ , and write  $m(x') = t \nabla K^L(x', t)$  for every  $x' \in \mathbb{R}^{n-1}$ . We have already shown in (3.78) that

$$\int_{\mathbb{R}^{n-1}} m(x') dx' = 0. \quad (4.115)$$

Also, by part (5) in Theorem 2.3 we have

$$\begin{aligned} \int_{B_t} |m(x')|^2 dx' &\leq C \int_{|x'| < t} \frac{t^2}{(t + |x'|)^{2n}} dx' \leq C \int_{|x'| < t} \frac{t^2}{t^{2n}} dx' \\ &= C t^{1-n} \leq C_0^2 |B_t|^{-1}, \end{aligned} \quad (4.116)$$

and, for every  $k \geq 1$ ,

$$\begin{aligned} \int_{2^k B_t \setminus 2^{k-1} B_t} |m(x')|^2 dx' &\leq C \int_{2^{k-1} t < |x'| < 2^k t} \frac{t^2}{(t + |x'|)^{2n}} dx' \\ &\leq C \int_{2^k B_t} \frac{t^2}{(2^k t)^{2n}} dx' \\ &= C 2^{-2k} (2^k t)^{1-n} \leq C_0^2 2^{-2k} |2^k B_t|^{-1}, \end{aligned} \quad (4.117)$$

for some constant  $C_0 \in (0, \infty)$  independent of  $k$ ,  $x'$ , and  $t$ . All these give that  $C_0^{-1}m$  is an  $(L^2(\mathbb{R}^{n-1}), 1)$ -molecule relative to  $B_t$  and (4.112) follows from the molecular characterization of  $H^1(\mathbb{R}^{n-1})$ , see [3].

In addition, for each  $x' \in \mathbb{R}^{n-1}$  fixed, the function  $C_0^{-1}m(x' - \cdot)$  is an  $(L^2(\mathbb{R}^{n-1}), 1)$ -molecule relative to  $B_{n-1}(x', t)$  and therefore belongs to  $H^1(\mathbb{R}^{n-1})$ . Hence, (4.114) follows from the definition of the weak-\* convergence.  $\square$

We now have all the ingredients to prove Proposition 4.1:

*Proof of Proposition 4.1.* Under the notation of Lemma 4.13, from (4.106) we know that the sequence  $\{[f_\varepsilon]\}_{0 < \varepsilon < 1}$  is bounded in the Banach space  $\widetilde{\text{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . By eventually passing to a subsequence, Alaoglu's theorem guarantees that there is no loss of generality in assuming that  $\{[f_\varepsilon]\}_{0 < \varepsilon < 1}$  converges weakly in  $\widetilde{\text{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  to some  $[g] \in \widetilde{\text{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  (where  $g \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ ) satisfying

$$\|[g]\|_{\widetilde{\text{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq C \|u\|_{**}, \quad (4.118)$$

for some finite constant  $C > 0$  which does not depend on  $u$ . Applying Lemma 4.14, for every  $(x', t) \in \mathbb{R}_+^n$  fixed we may conclude with the help of (4.69) that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \nabla[(P_t^L * f_\varepsilon)(x')] &= \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^{n-1}} (\nabla K^L)(x' - y', t) f_\varepsilon(y') dy' \\ &= \int_{\mathbb{R}^{n-1}} (\nabla K^L)(x' - y', t) g(y') dy' \\ &= \nabla[(P_t^L * g)(x')]. \end{aligned} \quad (4.119)$$

On the other hand, by Lemma 4.7 we have

$$\nabla u(x', t + \varepsilon) = \nabla u_\varepsilon(x', t) = \nabla[(P_t^L * f_\varepsilon)(x')], \quad \forall (x', t) \in \mathbb{R}_+^n. \quad (4.120)$$

Together, (4.119) and (4.120) give (keeping in mind part (a) in Lemma 4.6)

$$\nabla u(x', t) = \lim_{\varepsilon \rightarrow 0^+} \nabla u(x', t + \varepsilon) = \nabla [(P_t^L * g)(x')], \quad \forall (x', t) \in \mathbb{R}_+^n. \quad (4.121)$$

Consequently, there exists  $C \in \mathbb{C}^M$  with the property that

$$u(x', t) = (P_t^L * g)(x') + C, \quad \forall (x', t) \in \mathbb{R}_+^n. \quad (4.122)$$

Then  $f := g + C \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  satisfies (thanks to (4.118) and (1.19))

$$\|f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} = \|g\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} = \|[g]\|_{\widetilde{\text{BMO}}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq C \|u\|_{**}, \quad (4.123)$$

where  $C > 0$  is a finite constant which does not depend on  $u$ . Moreover, in light of (2.36), formula (4.122) becomes precisely (4.2).

Having established this, we may invoke part (7) in Theorem 2.3 (whose applicability is ensured by (2.24)) in order to conclude that  $u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}}$  exists at a.e. point in  $\mathbb{R}^{n-1}$  and, in fact,  $f = u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}}$ . To conclude the proof of Proposition 4.1 there remains to observe that (4.3) is a consequence of (4.123), (4.2), and (3.12).  $\square$

## 5. PROOFS OF THEOREMS 1.1, 1.2, 1.3, 1.4, 1.6, AND 1.8

We begin by presenting the proof of the Fatou type result stated in Theorem 1.2. The main ingredients involved are Proposition 3.1, Proposition 4.1, and Proposition 4.2.

*Proof of Theorem 1.2.* The implication in (1.34) is seen directly from Proposition 4.1. Proposition 4.1 also guarantees the right-to-left inclusion in (1.35). The left-to-right inclusion in (1.35) is a consequence of Proposition 3.1. Going further, it is clear from definitions that  $\text{LMO}(\mathbb{R}_+^n)$  is a linear space on which  $\|\cdot\|_{**}$  is a seminorm with null-space  $\mathbb{C}^M$ . The linear mapping in (1.37) is bounded (by the estimate in (1.34)), injective (by Proposition 4.2), and surjective (by Proposition 3.1). Moreover, another reference to the estimate in (1.34) shows that the inverse of the mapping (1.37) is also bounded. Given that  $\widetilde{\text{BMO}}(\mathbb{R}_+^n)$  is complete, it follows that the quotient space  $\text{LMO}(\mathbb{R}_+^n)/\mathbb{C}^M$  is also complete when equipped with  $\|\cdot\|_{**}$ .  $\square$

Anticipating the proof of Theorem 1.3, below we isolate a key result to the effect that any smooth null-solution of  $L$  satisfying a vanishing Carleson measure condition in the upper-half space converges vertically to its nontangential boundary trace in BMO.

**Lemma 5.1.** *Let  $L$  be an  $M \times M$  elliptic system with constant complex coefficients as in (1.1)-(1.2) and consider  $P^L$ , the associated Poisson kernel for  $L$  in  $\mathbb{R}_+^n$  from Theorem 2.3. Suppose  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$  satisfies  $Lu = 0$  in  $\mathbb{R}_+^n$  and  $\|u\|_{**} < \infty$  and use Theorem 1.2 to write*

$$f := u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M). \quad (5.1)$$

For each number  $\varepsilon > 0$ , define  $u_\varepsilon(x', t) := u(x', t + \varepsilon)$  for every  $(x', t) \in \mathbb{R}_+^n$  and consider  $f_\varepsilon := u_\varepsilon|_{\partial \mathbb{R}_+^n} \in \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  (see Lemma 4.13). Then

$$\left. \begin{array}{l} |\nabla u(x', t)|^2 t \, dx' dt \text{ is a vanishing} \\ \text{Carleson measure in } \mathbb{R}_+^n \end{array} \right\} \implies \lim_{\varepsilon \rightarrow 0^+} \|f_\varepsilon - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} = 0. \quad (5.2)$$

*Proof.* By Theorem 1.2 we have  $u(x', t) = (P_t^L * f)(x')$  for every  $(x', t) \in \mathbb{R}_+^n$ . Also, Lemma 4.7 implies  $u_\varepsilon(x', t) = (P_t^L * f_\varepsilon)(x')$  for every  $(x', t) \in \mathbb{R}_+^n$  and each  $\varepsilon > 0$ . To proceed, for every  $(x', t) \in \mathbb{R}_+^n$  set

$$v_\varepsilon(x', t) := (P_t^L * (f_\varepsilon - f))(x', t) = (P_t^L * f)(x', t) - (P_t^L * f_\varepsilon)(x', t)$$

$$= u_\varepsilon(x', t) - u(x', t). \quad (5.3)$$

Given that for each parameter  $\varepsilon > 0$  the function  $v_\varepsilon$  satisfies the hypotheses of Theorem 1.2 and  $v_\varepsilon|_{\partial\mathbb{R}_+^n} = f_\varepsilon - f \in \text{BMO}(\mathbb{R}^{n-1})$  it follows that

$$\|f_\varepsilon - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq C \|v_\varepsilon\|_{**} = C \|u_\varepsilon - u\|_{**} \quad (5.4)$$

for every  $\varepsilon > 0$ . Hence, to complete the proof of (5.2) it suffices to show that

$$\lim_{\varepsilon \rightarrow 0^+} \|u_\varepsilon - u\|_{**} = 0. \quad (5.5)$$

To this end, for each  $r > 0$  we introduce

$$\|u\|_{**,r} := \sup_{Q \subset \mathbb{R}^{n-1}, \ell(Q) \leq r} \left( \int_0^{\ell(Q)} \int_Q |\nabla u(x', t)|^2 t \, dx' dt \right)^{\frac{1}{2}}. \quad (5.6)$$

Note that

$$\|u\|_{**,r} \leq \|u\|_{**,s} \leq \|u\|_{**} \quad \text{if } r \leq s, \quad (5.7)$$

and the fact that  $|\nabla u(x', t)|^2 t \, dx' dt$  is a vanishing Carleson measure in  $\mathbb{R}_+^n$  (recall (1.11)) may be rephrased as

$$\|u\|_{**,r} \rightarrow 0 \quad \text{as } r \rightarrow 0^+. \quad (5.8)$$

We now make the claim that there exists a constant  $C = C(n, L) \in (0, \infty)$  such that

$$\|u - u_\varepsilon\|_{**} \leq C (\|u\|_{**,4 \max\{r, \varepsilon\}} + \|u\|_{**} \min\{\varepsilon/r, 1\}), \quad \forall r, \varepsilon \in (0, \infty). \quad (5.9)$$

Assume the claim for now and based on (5.9), for every  $0 < r < \infty$ , we may write

$$\begin{aligned} 0 &\leq \limsup_{\varepsilon \rightarrow 0^+} \|u - u_\varepsilon\|_{**} \\ &\leq C \limsup_{\varepsilon \rightarrow 0^+} \|u\|_{**,4 \max\{r, \varepsilon\}} + C \|u\|_{**} \limsup_{\varepsilon \rightarrow 0^+} [\min\{\varepsilon/r, 1\}] \\ &= C \|u\|_{**,4r}. \end{aligned} \quad (5.10)$$

Recalling now (5.8), we may further take the limit as  $r \rightarrow 0^+$  in the resulting inequality in (5.10) and conclude that  $\limsup_{\varepsilon \rightarrow 0^+} \|u - u_\varepsilon\|_{**} = 0$ . This proves (5.5) as wanted.

To finish the proof of the lemma we are left with showing the claim. To do so, we first note that in light of the notation in (5.6), the reasoning in (4.9) (corresponding to  $|\alpha| = 1$ ) yields

$$\begin{aligned} t |\nabla u(x', t)| &\leq C \left( \frac{1}{|Q_{x'}|} \int_{t/2}^{3t/2} \int_{Q_{x'}} |\nabla u(y', s)|^2 s \, dy' ds \right)^{\frac{1}{2}} \\ &\leq C \|u\|_{**,2t} \end{aligned} \quad (5.11)$$

for each  $(x', t) \in \mathbb{R}_+^n$ , where  $Q_{x'}$  denotes the cube in  $\mathbb{R}^{n-1}$  centered at  $x'$  with side-length  $t$ .

Next, fix a cube  $Q \subset \mathbb{R}^{n-1}$  and numbers  $r, \varepsilon \in (0, \infty)$  and proceed by analyzing the following possible three cases.

**Case 1:**  $\ell(Q) \leq \varepsilon$ . Under this assumption, recalling also (5.11) and (5.7), we obtain

$$\begin{aligned} &\left( \int_0^{\ell(Q)} \int_Q |\nabla u_\varepsilon(x', t) - \nabla u(x', t)|^2 t \, dx' dt \right)^{\frac{1}{2}} \\ &\leq \left( \int_\varepsilon^{\ell(Q)+\varepsilon} \int_Q |\nabla u(x', t)|^2 t \, dx' dt \right)^{\frac{1}{2}} + \left( \int_0^{\ell(Q)} \int_Q |\nabla u(x', t)|^2 t \, dx' dt \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq \left( \int_{\varepsilon}^{2\varepsilon} \int_Q |\nabla u(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} + \|u\|_{**, \varepsilon} \\
&\leq C \left( \int_{\varepsilon}^{2\varepsilon} \frac{\|u\|_{**, 2t}^2}{t} dt \right)^{\frac{1}{2}} + \|u\|_{**, 4\varepsilon} \leq C \|u\|_{**, 4\varepsilon} \\
&\leq C \|u\|_{**, 4 \max\{r, \varepsilon\}},
\end{aligned} \tag{5.12}$$

for some constant  $C = C(n, L) \in (0, \infty)$  independent of  $u$ ,  $\varepsilon$ , and  $r$ .

**Case 2:**  $\varepsilon < \ell(Q) \leq r$ . Note that in this case  $r = \max\{r, \varepsilon\}$  and using again (5.11) and (5.7) we have

$$\begin{aligned}
&\left( \int_0^{\ell(Q)} \int_Q |\nabla u_{\varepsilon}(x', t) - \nabla u(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} \\
&\leq \left( \int_{\varepsilon}^{\ell(Q)+\varepsilon} \int_Q |\nabla u(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} + \|u\|_{**, r} \\
&\leq \left( \int_0^{2\ell(Q)} \int_Q |\nabla u(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} + \|u\|_{**, \max\{r, \varepsilon\}} \\
&\leq C \|u\|_{**, 4 \max\{r, \varepsilon\}},
\end{aligned} \tag{5.13}$$

for some constant  $C = C(n, L) \in (0, \infty)$  independent of  $u$ ,  $\varepsilon$ , and  $r$ .

**Case 3:**  $\ell(Q) > \max\{r, \varepsilon\}$ . In this case, set  $\eta := \max\{r, \varepsilon\}$  and write

$$\begin{aligned}
&\left( \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\nabla u_{\varepsilon}(x', t) - \nabla u(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} \\
&\leq \left( \frac{1}{|Q|} \int_0^{\eta} \int_Q |\nabla u_{\varepsilon}(x', t) - \nabla u(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} \\
&\quad + \left( \frac{1}{|Q|} \int_{\eta}^{\ell(Q)} \int_Q |\nabla u_{\varepsilon}(x', t) - \nabla u(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} =: I + II.
\end{aligned} \tag{5.14}$$

To analyze  $I$ , let  $k_0$  be the nonnegative integer such that  $2^{k_0} \eta < \ell(Q) \leq 2^{k_0+1} \eta$ . Also, consider  $\{Q_j\}_{j \in \mathbb{N}}$ , the collection of subcubes of  $Q$  with pairwise disjoint interiors, satisfying  $\ell(Q_j) = 2^{-k_0} \ell(Q)$  for each  $j \in \mathbb{N}$  and  $\bigcup_{j \in \mathbb{N}} Q_j = Q$ . Then  $\ell(Q_j) \in (\eta, 2\eta]$  for every  $j \in \mathbb{N}$ .

Bearing this in mind and using the fact that  $\varepsilon \leq \eta$ , we may then estimate

$$\begin{aligned}
I &\leq \left( \frac{1}{|Q|} \int_{\varepsilon}^{\eta+\varepsilon} \int_Q |\nabla u(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} + \left( \frac{1}{|Q|} \int_0^{\eta} \int_Q |\nabla u(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} \\
&\leq 2 \left( \frac{1}{|Q|} \int_0^{2\eta} \int_Q |\nabla u(x', t)|^2 t dx' dt \right)^{\frac{1}{2}} \\
&\leq 2 \left( \frac{1}{|Q|} \sum_{j \in \mathbb{N}} \int_0^{2\ell(Q_j)} \int_{Q_j} |\nabla u(x', t)|^2 t dx' dt \right)^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
&\leq 2 \left( \frac{1}{|Q|} \sum_{j \in \mathbb{N}} \|u\|_{**, 2\ell(Q_j)}^2 |2Q_j| \right)^{\frac{1}{2}} \leq 2^{\frac{n+1}{2}} \|u\|_{**, 4\eta} \\
&= 2^{\frac{n+1}{2}} \|u\|_{**, 4 \max\{r, \varepsilon\}}.
\end{aligned} \tag{5.15}$$

Up to this point our treatment involved estimating  $u_\varepsilon$  and  $u$  separately, without exploiting any potential cancellations generated by the fact that we are dealing with their difference. However, in the task of estimating  $II$ , having the difference  $u_\varepsilon - u$  plays a crucial role, as seen next. Given  $(x', t) \in \mathbb{R}_+^n$  we write  $R_x$  for the cube in  $\mathbb{R}^n$  centered at  $x$  with side-length  $t/2$ , and let  $Q_{x'}$  be the cube in  $\mathbb{R}^{n-1}$  centered at  $x'$  with side-length  $t$ . From Lemma 4.3 we conclude that

$$|\nabla^2 u(x', t)| \leq C \|u\|_{**} t^{-2}. \tag{5.16}$$

In light of this, the Mean Value Theorem implies that for some  $\theta \in (0, 1)$  there holds

$$\begin{aligned}
|\nabla u_\varepsilon(x', t) - \nabla u(x', t)| &= |\nabla u(x', t + \varepsilon) - \nabla u(x', t)| \leq \varepsilon |\nabla^2 u(x', t + \theta\varepsilon)| \\
&\leq C \varepsilon \|u\|_{**} (t + \theta\varepsilon)^{-2} \\
&\leq C \varepsilon \|u\|_{**} t^{-2}.
\end{aligned} \tag{5.17}$$

Having established (5.17), we may turn to estimating  $II$  as follows

$$\begin{aligned}
II &= \left( \frac{1}{|Q|} \int_\eta^{\ell(Q)} \int_Q |\nabla u_\varepsilon(x', t) - \nabla u(x', t)|^2 t \, dx' dt \right)^{\frac{1}{2}} \\
&\leq C \varepsilon \|u\|_{**} \left( \int_\eta^{\ell(Q)} t^{-3} dt \right)^{\frac{1}{2}} \leq C \varepsilon \|u\|_{**} \eta^{-1} \\
&= C \varepsilon \|u\|_{**} (\max\{r, \varepsilon\})^{-1} = C \|u\|_{**} \min\{\varepsilon/r, 1\}.
\end{aligned} \tag{5.18}$$

In concert, (5.14), (5.15), and (5.18), allow us to conclude that, under the current assumption  $\ell(Q) > \max\{r, \varepsilon\}$ , we have

$$\begin{aligned}
&\left( \int_0^{\ell(Q)} \int_Q |\nabla u_\varepsilon(x', t) - \nabla u(x', t)|^2 t \, dx' dt \right)^{\frac{1}{2}} \\
&\leq C (\|u\|_{**, 4 \max\{r, \varepsilon\}} + \|u\|_{**} \min\{\varepsilon/r, 1\}).
\end{aligned} \tag{5.19}$$

Combining (5.12), (5.13), and (5.19), we obtain that the estimate in (5.19) actually holds for arbitrary cubes  $Q$  in  $\mathbb{R}^{n-1}$ . In turn, the latter yields (5.9) after taking the supremum over all cubes  $Q$  in  $\mathbb{R}^{n-1}$ . With this the proof of the lemma is completed.  $\square$

Having proved the convergence result in Lemma 5.1, we are now prepared to present the proof of Theorem 1.3.

*Proof of Theorem 1.3.* We start by noticing that since  $u$  satisfies the conditions in (1.38), the conclusions in (1.34) hold. Hence if we set  $f := u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}}$ , we have that  $f$  exists almost everywhere in  $\mathbb{R}^{n-1}$  and belongs to  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . To proceed in showing that  $f \in \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ , for each  $\varepsilon > 0$  define  $u_\varepsilon(x', t) := u(x', t + \varepsilon)$  for every  $(x', t) \in \mathbb{R}_+^n$ , and  $f_\varepsilon(x') := u(x', \varepsilon)$  for every  $x' \in \mathbb{R}^{n-1}$ . Then from Lemma 4.13 and part (d) in Lemma 4.6 we obtain  $f_\varepsilon \in$

$\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M) \cap \mathcal{C}^\infty(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . In addition, for every  $\varepsilon > 0$ , based on Lemma 4.3 we obtain

$$\sup_{x' \in \mathbb{R}^{n-1}} |\nabla_{x'} f_\varepsilon(x')| = \sup_{x' \in \mathbb{R}^{n-1}} |\nabla_{x'} u(x', \varepsilon)| \leq C \varepsilon^{-1} \|u\|_{**} < \infty. \quad (5.20)$$

Fix  $r \in (0, \infty)$  and let  $Q \subset \mathbb{R}^{n-1}$  be a cube in  $\mathbb{R}_+^n$  with  $\ell(Q) \leq r$ . Then using (5.20) we may estimate

$$\int_Q |f(x') - f_Q| dx' \leq \int_Q |(f - f_\varepsilon)(x') - (f - f_\varepsilon)_Q| dx' + \int_Q |f_\varepsilon(x') - (f_\varepsilon)_Q| dx' \quad (5.21)$$

$$\leq \|f_\varepsilon - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} + \sup_{x' \in \mathbb{R}^{n-1}} |\nabla_{x'} f_\varepsilon(x')| \sqrt{n-1} \ell(Q) \quad (5.22)$$

$$\leq \|f_\varepsilon - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} + C r \varepsilon^{-1} \|u\|_{**}. \quad (5.23)$$

Hence,

$$\sup_{Q \subset \mathbb{R}^{n-1}, \ell(Q) \leq r} \int_Q |f(x') - f_Q| dx' \leq \|f_\varepsilon - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} + C r \varepsilon^{-1} \|u\|_{**}. \quad (5.24)$$

Letting  $r \rightarrow 0^+$  first, then sending  $\varepsilon \rightarrow 0^+$  in (5.24) and recalling that since  $|\nabla u(x', t)|^2 t dx' dt$  is a vanishing Carleson measure in  $\mathbb{R}_+^n$  implication (5.2) holds, we conclude that

$$\lim_{r \rightarrow 0^+} \sup_{Q \subset \mathbb{R}^{n-1}, \ell(Q) \leq r} \int_Q |f(x') - f_Q| dx' = 0. \quad (5.25)$$

Hence,  $f \in \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ , as wanted.

To complete the proof, there remains to establish (1.40). However, the right-to-left inclusion follows from (1.39), while the opposite inclusion is a consequence of Proposition 3.1.  $\square$

Having dealt with the Fatou type results from Theorems 1.2-1.3, we turn our attention to the proof of Theorem 1.1.

*Proof of Theorem 1.1.* The fact that the function  $u$  defined in (1.25) solves the BMO-Dirichlet boundary value problem (1.24) follows from Proposition 3.1. By Proposition 4.2, this is the only solution of (1.24). Next, the double estimate in (1.26) is a direct consequence of (1.25) and (4.3). The uniform BMO estimate in (1.27) is seen straight from Lemma 4.13.

Moving on, the left-pointing implication in (1.28) follows from Lemma 5.1. For the opposite implication, invoke part (d) in Lemma 4.6 together with (1.27) to conclude that  $f$  is the limit in  $\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  of the sequence  $\{u(\cdot, \varepsilon)\}_{\varepsilon > 0} \subset \text{UC}(\mathbb{R}^{n-1}, \mathbb{C}^M) \cap \text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . This places  $f$  in  $\text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  (cf. (1.22)). Having established this, the last property in Proposition 3.1 gives that  $|\nabla u(x', t)|^2 t dx' dt$  is a vanishing Carleson measure in  $\mathbb{R}_+^n$ . Going further, the equivalence in (1.29) is seen from (1.39) and the last part in Proposition 3.1.

As regards the equivalence in (1.30), let us first observe that, as is apparent from its definition in (1.12), the seminorm  $\|\cdot\|_{**}$  is invariant to horizontal translations. That is, for every  $u \in \mathcal{C}^1(\mathbb{R}_+^n, \mathbb{C}^M)$  we have

$$\|\tau_{(z', 0)} u\|_{**} = \|u\|_{**} \quad \text{for every } z' \in \mathbb{R}^{n-1}. \quad (5.26)$$

Given  $f \in \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ , the right-pointing implication in (1.29) ensures that

$$|\nabla u(x', t)|^2 t dx' dt \text{ is a vanishing Carleson measure in } \mathbb{R}_+^n. \quad (5.27)$$

For each  $z = (z', s) \in \mathbb{R}_+^n$  we may write, thanks to (5.26) and the estimate in (1.34),

$$\begin{aligned} \|\tau_z u - u\|_{**} &\leq \|\tau_z u - \tau_{(z', 0)} u\|_{**} + \|\tau_{(z', 0)} u - u\|_{**} \\ &= \|\tau_{(0, s)} u - u\|_{**} + \|\tau_{(z', 0)} u - u\|_{**} \end{aligned}$$



$$\leq C\|u(\cdot, s) - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} + C\|\tau_{z'}f - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)}, \quad (5.28)$$

for some constant  $C = C(n, L) \in (0, \infty)$ . In light of (5.27), the left-pointing implication in (1.28), and (1.23), we then conclude from (5.28) that

$$\lim_{\mathbb{R}_+^n \ni z \rightarrow 0} \|\tau_z u - u\|_{**} = 0, \quad (5.29)$$

as wanted. Conversely, suppose now that (5.29) holds. Specializing this to the case when  $z := (0', \varepsilon)$  with  $\varepsilon > 0$  then yields, on account of the estimate in (1.34),

$$\|u(\cdot, \varepsilon) - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq C\|\tau_{(0', \varepsilon)}u - u\|_{**} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0^+. \quad (5.30)$$

Hence,  $\|u(\cdot, \varepsilon) - f\|_{\text{BMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)} \rightarrow 0$  as  $\varepsilon \rightarrow 0^+$  which, by virtue of (1.28)-(1.29), implies that  $f \in \text{VMO}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . This finishes the proofs of the equivalences in part (iv) of the statement.

Finally, all claims about the VMO-Dirichlet boundary value problem (1.31) are direct consequences of what we have proved up to this point.  $\square$

Going further, we present the proof of the quantitative characterization of VMO from Theorem 1.4.

*Proof of Theorem 1.4.* We shall establish all claims stated with  $n - 1$  in place on  $n$ . Fix a modulus of continuity  $\Upsilon$  satisfying  $\Upsilon_{\#} \leq C\Upsilon$  on  $[0, \infty)$  for some finite constant  $C > 0$ . This implies that

$$\mathcal{C}^{\Upsilon_{\#}}(\mathbb{R}^{n-1}) \subseteq \mathcal{C}^{\Upsilon}(\mathbb{R}^{n-1}). \quad (5.31)$$

Consider next an arbitrary function  $f \in \text{VMO}(\mathbb{R}^{n-1})$  and define  $u \in \mathcal{C}^{\infty}(\mathbb{R}_+^n)$  by setting  $u(x', t) := (P_t^{\Delta} * f)(x')$  for  $(x', t) \in \mathbb{R}_+^n$ . Then from item (d) in Lemma 4.6, Theorem 1.1 part (iii), and (1.32) we conclude that the sequence of functions  $\{f_{\varepsilon}\}_{\varepsilon > 0}$  defined for every  $\varepsilon > 0$  by  $f_{\varepsilon} := u(\cdot, \varepsilon)$  in  $\mathbb{R}^{n-1}$  satisfies, for each  $\varepsilon > 0$ ,

$$\begin{aligned} f_{\varepsilon} &\in \mathcal{C}^{\Upsilon}(\mathbb{R}^{n-1}) \cap \mathcal{C}^{\infty}(\mathbb{R}^{n-1}) \cap \text{BMO}(\mathbb{R}^{n-1}) \text{ and} \\ \partial^{\alpha'} f &\in \mathcal{C}^{\Upsilon}(\mathbb{R}^{n-1}) \cap L^{\infty}(\mathbb{R}^{n-1}) \text{ for every } \alpha' \in \mathbb{N}_0^{n-1} \text{ with } |\alpha'| \geq 1, \end{aligned} \quad (5.32)$$

as well as

$$\|f - f_{\varepsilon}\|_{\text{BMO}(\mathbb{R}^{n-1})} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0^+. \quad (5.33)$$

This establishes (1.45), as well as the stronger claim made in (1.46).  $\square$

Finally, here is the proof of the negative result stated in Theorem 1.6.

*Proof of Theorem 1.6.* From [5, Proposition 9, p. 1208] we know that there exists  $f \in \mathcal{C}^{\infty}(\mathbb{R}^n)$  such that

$$\partial^{\alpha} f \in \text{BMO}(\mathbb{R}^n), \quad \forall \alpha \in \mathbb{N}_0^n, \quad (5.34)$$

and

$$\inf \{ \|f - g\|_{\text{BMO}(\mathbb{R}^n)} : g \in L^{\infty}(\mathbb{R}^n) \} > 0. \quad (5.35)$$

In concert with [5, Lemme 6, p. 1211], property (5.34) (used for multi-indices  $\alpha \in \mathbb{N}_0^n$  with  $|\alpha| = 1$ ) entails  $f \in \text{UC}(\mathbb{R}^n)$ . By once again using (5.34) (with  $|\alpha| = 0$ ), this proves that  $f \in \text{UC}(\mathbb{R}^n) \cap \text{BMO}(\mathbb{R}^n)$ , hence  $f \in \text{VMO}(\mathbb{R}^n)$ . On the other hand, (5.35) implies that  $f$  does not belong to the closure of  $L^{\infty}(\mathbb{R}^n)$  in  $\text{BMO}(\mathbb{R}^n)$ , hence also  $f$  does not belong to the closure of  $\text{UC}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$  in  $\text{BMO}(\mathbb{R}^n)$ . Ultimately, this proves that the space  $\text{UC}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$  is not dense in  $\text{VMO}(\mathbb{R}^n)$ .  $\square$

We conclude this section by providing the proof of Theorem 1.8.

*Proof of Theorem 1.8.* First note that condition (1.59) implies that  $\varphi$  is continuous on  $\mathbb{R}^n \setminus \{0\}$ . As such,  $\varphi$  is a Lebesgue measurable function  $\mathbb{R}^n$  which, in turn, ensures that condition (1.60) is meaningful.

To proceed, observe that if  $f \in L^1\left(\mathbb{R}^n, \frac{dx}{1+|x|^{n+\varepsilon}}\right)^M$  then for each  $x \in \mathbb{R}^n$  we have

$$\begin{aligned} \int_{\mathbb{R}^n} |f(y)| |\varphi(x-y)| dy &\leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{(1+|y|)^{n+\varepsilon}} \cdot \frac{(1+|y|)^{n+\varepsilon}}{(1+|x-y|)^{n+\varepsilon}} dy \\ &\leq C(1+|x|)^{n+\varepsilon} \int_{\mathbb{R}^n} \frac{|f(y)|}{(1+|y|)^{n+\varepsilon}} dy < \infty. \end{aligned} \quad (5.36)$$

In light of (2.24) (used here with  $n+1$  in place of  $n$ ), this implies that for every  $t > 0$  the convolution  $\varphi_t * f$  is well-defined via an absolutely convergent integral whenever the function  $f$  belongs to  $\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)$ . In particular, this is the case whenever  $f \in \text{VMO}(\mathbb{R}^n, \mathbb{C}^M)$ .

Next, fix  $t > 0$  and define

$$T_t f := \varphi_t * f \quad \text{for every } f \in \text{BMO}(\mathbb{R}^n, \mathbb{C}^M). \quad (5.37)$$

We first claim that there exists some constant  $C \in (0, \infty)$  independent of  $t$  such that

$$\|T_t f\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)} \leq C \|f\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)} \quad \text{for all } f \in \text{BMO}(\mathbb{R}^n, \mathbb{C}^M). \quad (5.38)$$

To prove this claim, fix  $f \in \text{BMO}(\mathbb{R}^n, \mathbb{C}^M)$  and an arbitrary cube  $Q$  in  $\mathbb{R}^n$  with center  $x_Q$ , then decompose

$$f = (f - f_Q) \mathbf{1}_{\lambda Q} + (f - f_Q) \mathbf{1}_{\mathbb{R}^n \setminus \lambda Q} + f_Q, \quad \text{where } \lambda := 2\sqrt{n}. \quad (5.39)$$

Thus, using (1.60) we have

$$(T_t f)(x) = T_t[(f - f_Q) \mathbf{1}_{\lambda Q}](x) + T_t[(f - f_Q) \mathbf{1}_{\mathbb{R}^n \setminus (\lambda Q)}](x) + f_Q \quad \forall x \in \mathbb{R}^n, \quad (5.40)$$

and if we set

$$c_Q := T_t[(f - f_Q) \mathbf{1}_{\mathbb{R}^n \setminus (\lambda Q)}](x_Q) + f_Q \in \mathbb{C}^M \quad (5.41)$$

it follows that

$$\begin{aligned} \int_Q |(T_t f)(x) - c_Q| dx &\leq \int_Q |T_t[(f - f_Q) \mathbf{1}_{\lambda Q}](x)| dx \\ &\quad + \int_Q |T_t[(f - f_Q) \mathbf{1}_{\mathbb{R}^n \setminus \lambda Q}](x) - T_t[(f - f_Q) \mathbf{1}_{\mathbb{R}^n \setminus \lambda Q}](x_Q)| dx \\ &=: I + II. \end{aligned} \quad (5.42)$$

Since  $f \in \text{BMO}(\mathbb{R}^n, \mathbb{C}^M)$  we have  $(f - f_Q) \mathbf{1}_{\lambda Q} \in L^1(\mathbb{R}^n, \mathbb{C}^M)$ . On the other hand, assumption (1.58) implies that  $T_t$  is bounded in  $L^1(\mathbb{R}^n, \mathbb{C}^M)$  uniformly in  $t$ . In concert with (2.8), this permits us to estimate

$$\begin{aligned} I &= \frac{1}{|Q|} \|T_t[(f - f_Q) \mathbf{1}_{\lambda Q}]\|_{L^1(\mathbb{R}^n, \mathbb{C}^M)} \\ &\leq \frac{C}{|Q|} \|(f - f_Q) \mathbf{1}_{\lambda Q}\|_{L^1(\mathbb{R}^n, \mathbb{C}^M)} \leq C \|f\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)}, \end{aligned} \quad (5.43)$$

for some  $C \in (0, \infty)$  independent of  $f$ ,  $Q$ , and  $t$ . To treat  $II$ , first we derive a pointwise estimate. For each  $x \in Q$  we have

$$\begin{aligned} &|T_t[(f - f_Q) \mathbf{1}_{\mathbb{R}^n \setminus \lambda Q}](x) - T_t[(f - f_Q) \mathbf{1}_{\mathbb{R}^n \setminus \lambda Q}](x_Q)| \\ &\leq t^{-n} \int_{\mathbb{R}^n \setminus \lambda Q} |f(y) - f_Q| \left| \varphi\left(\frac{x-y}{t}\right) - \varphi\left(\frac{x_Q-y}{t}\right) \right| dy. \end{aligned} \quad (5.44)$$

Next, pick some arbitrary  $x \in Q$  and  $y \in \mathbb{R}^n \setminus \lambda Q$ , then consider  $z := (x_Q - y)/t \in \mathbb{R}^n \setminus \{0\}$  and  $h := (x - x_Q)/t \in \mathbb{R}^n$ . Since in view of the choice of  $\lambda$  in (5.39) we have

$$|h| \leq \frac{\sqrt{n}\ell(Q)}{2t} = \frac{\lambda\ell(Q)}{4t} \leq \frac{|z|}{2}, \quad (5.45)$$

it follows from (1.59) that

$$\begin{aligned} \left| \varphi\left(\frac{x-y}{t}\right) - \varphi\left(\frac{x_Q-y}{t}\right) \right| &= |\varphi(z+h) - \varphi(z)| \leq \frac{C|h|^\varepsilon}{|z|^{n+\varepsilon}} \\ &\leq \frac{C\ell(Q)^\varepsilon t^n}{|y-x_Q|^{n+\varepsilon}} \leq \frac{C\ell(Q)^\varepsilon t^n}{(\ell(Q) + |y-x_Q|)^{n+\varepsilon}}. \end{aligned} \quad (5.46)$$

Combining (5.44)-(5.46) with (2.21) (used here with  $n+1$  in place of  $n$ ) and part (c) in Lemma 2.1 it follows that

$$\begin{aligned} &|T_t[(f - f_Q)\mathbf{1}_{\mathbb{R}^n \setminus \lambda Q}](x) - T_t[(f - f_Q)\mathbf{1}_{\mathbb{R}^n \setminus \lambda Q}](x_Q)| \\ &\leq C\ell(Q)^\varepsilon \int_{\mathbb{R}^n} \frac{|f(y) - f_Q|}{(\ell(Q) + |y - x_Q|)^{n+\varepsilon}} dy \\ &\leq C \int_1^\infty \text{osc}_1(f; \lambda\ell(Q)) \frac{d\lambda}{\lambda^{1+\varepsilon}} \\ &\leq C\|f\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)}, \quad \forall x \in Q, \end{aligned} \quad (5.47)$$

where  $C \in (0, \infty)$  is independent of  $f, Q$  and  $t$ . From (5.47) and (5.42) we obtain

$$II \leq C\|f\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)} \quad (5.48)$$

for some  $C \in (0, \infty)$  independent of  $f, Q$ , and  $t$ . In concert, (5.42), (5.43), and (5.48) yield

$$\int_Q |(T_t f)(x) - c_Q| dx \leq C\|f\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)} \quad (5.49)$$

with  $c_Q \in \mathbb{C}^M$  as in (5.41). In view of (2.9), this ultimately implies the claim in (5.38).

The second claim we make is that there exists some constant  $C \in (0, \infty)$  with the property that for every  $t > 0$  and every  $\eta \in (0, \varepsilon)$  there holds

$$\|T_t g - g\|_{L^\infty(\mathbb{R}^n, \mathbb{C}^M)} \leq C t^\eta \|g\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}^n, \mathbb{C}^M)} \quad \text{for all } g \in \dot{\mathcal{C}}^\eta(\mathbb{R}^n, \mathbb{C}^M). \quad (5.50)$$

To prove this claim, fix  $t > 0$ ,  $\eta \in (0, \varepsilon)$ ,  $g \in \dot{\mathcal{C}}^\eta(\mathbb{R}^n, \mathbb{C}^M)$ , and for  $x \in \mathbb{R}^n$  arbitrary estimate

$$\begin{aligned} |(T_t g)(x) - g(x)| &\leq \int_{\mathbb{R}^n} |g(x-y) - g(x)| |\varphi_t(y)| dy \\ &\leq t^\eta \|g\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}^n, \mathbb{C}^M)} \int_{\mathbb{R}^n} \frac{|y|^\eta}{t^\eta} |\varphi_t(y)| dy \\ &\leq t^\eta \|g\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}^n, \mathbb{C}^M)} \int_{\mathbb{R}^n} |z|^\eta (1 + |z|)^{-n-\varepsilon} dz \\ &\leq C t^\eta \|g\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}^n, \mathbb{C}^M)}, \end{aligned} \quad (5.51)$$

for some constant  $C = C(\varepsilon, \eta, n, \varphi) \in (0, \infty)$  independent of  $t$  and  $g$ . The first inequality in (5.51) relies on (1.60), for the third one we have used (1.58) and the change of variables  $z = y/t$ , while the last one is a consequence of having  $\eta \in (0, \varepsilon)$ .

Here is the argument involved in the endgame of the proof of Theorem 1.8. Fix  $\eta \in (0, \varepsilon)$  and given  $f \in \text{VMO}(\mathbb{R}^n, \mathbb{C}^M)$  pick  $g \in \mathcal{C}^\eta(\mathbb{R}^n, \mathbb{C}^M) \cap \text{BMO}(\mathbb{R}^n, \mathbb{C}^M)$ . Then for each  $t > 0$ , we use (5.38) and (5.50) to estimate

$$\begin{aligned} \|T_t f - f\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)} &\leq \|T_t(f - g)\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)} + \|T_t g - g\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)} \\ &\quad + \|g - f\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)} \\ &\leq C\|g - f\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)} + 2\|T_t g - g\|_{L^\infty(\mathbb{R}^n, \mathbb{C}^M)} \\ &\leq C\|g - f\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)} + Ct^\eta \|g\|_{\mathcal{C}^\eta(\mathbb{R}^n, \mathbb{C}^M)}. \end{aligned} \quad (5.52)$$

Thus,

$$\limsup_{t \rightarrow 0^+} \|T_t f - f\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)} \leq C\|g - f\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)}. \quad (5.53)$$

Now (1.61) follows from (5.53) in light of the density result recorded in (1.49).

To prove the very last claim in the statement of Theorem 1.8, let  $\varphi \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{C}^{M \times M})$  be a function satisfying (1.62). Then for each  $x \in \mathbb{R}^n \setminus \{0\}$  and  $h \in \mathbb{R}^n$  with  $|h| < |x|/2$  the Mean Value Theorem permits us to estimate

$$\begin{aligned} |\varphi(x+h) - \varphi(x)| &\leq |h| \sup_{\xi \in [x, x+h]} |(\nabla \varphi)(\xi)| \\ &\leq C|h| \sup_{\xi \in [x, x+h]} (1 + |\xi|)^{-n-1} \leq \frac{C|h|}{|x|^{n+1}}. \end{aligned} \quad (5.54)$$

Hence, both (1.58) and (1.59) hold with  $\varepsilon = 1$  in this case, so the left-pointing implication in (1.63) is a consequence of (1.61).

As regards the right-pointing implication in (1.63), let us first observe that from (1.60) and (1.62) we have

$$\int_{\mathbb{R}^n} (\partial_j \varphi)((x-y)/t) dy = 0, \quad \forall x \in \mathbb{R}^n, \forall j \in \{1, \dots, n\}. \quad (5.55)$$

Next, given a function  $f \in \text{BMO}(\mathbb{R}^n, \mathbb{C}^M)$ , fix  $x \in \mathbb{R}^n$  and  $t > 0$  arbitrary and denote by  $Q_{x,t}$  the cube in  $\mathbb{R}^n$  centered at  $x$  and of sidelength  $t$ . As usual, abbreviate  $f_{Q_{x,t}} := \int_{Q_{x,t}} f(y) dy$ . On account of (5.55), (1.62), (2.21) (used here with  $\varepsilon = 1$  and  $n$  in place of  $n-1$ ), and (2.15) (used with  $p = 1$  and  $n$  in place of  $n-1$ ), for each  $j \in \{1, \dots, n\}$  we may then estimate

$$\begin{aligned} |\partial_j(\varphi_t * f)(x)| &= t^{-n-1} \left| \int_{\mathbb{R}^n} (\partial_j \varphi)\left(\frac{x-y}{t}\right) f(y) dy \right| \\ &= t^{-n-1} \left| \int_{\mathbb{R}^n} (\partial_j \varphi)\left(\frac{x-y}{t}\right) [f(y) - f_{Q_{x,t}}] dy \right| \\ &\leq C \int_{\mathbb{R}^n} \frac{|f(y) - f_{Q_{x,t}}|}{[t + |x-y|]^{n+1}} dy \leq Ct^{-1} \|f\|_{\text{BMO}(\mathbb{R}^n, \mathbb{C}^M)}, \end{aligned} \quad (5.56)$$

for some constant  $C \in (0, \infty)$  independent of  $f, x, t$ . In concert with (5.38), this proves that

$$\varphi_t * f \in \text{BMO}(\mathbb{R}^n, \mathbb{C}^M) \cap \text{Lip}(\mathbb{R}^n, \mathbb{C}^M) \quad \text{for each } t > 0. \quad (5.57)$$

With this in hand, the right-pointing implication in (1.63) readily follows (compare with (1.51)), finishing the proof of Theorem 1.8.  $\square$

## 6. PROOF OF THE WELL-POSEDNESS OF THE MORREY-CAMPANATO-DIRICHLET PROBLEM

This section is devoted to presenting the proof of Theorem 1.9. Throughout fix  $p, q \in [1, \infty)$ . We divide the proof into several steps, the starting point being the following claim:

**Step 1.** *There exists a constant  $C = C(n, L, \eta) \in (0, \infty)$  such that if  $f \in \mathcal{E}^{\eta, p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$  then the function  $u$  given at every point  $(x', t) \in \mathbb{R}_+^n$  by  $u(x', t) := (P_t^L * f)(x')$  is well-defined (via an absolutely convergent integral) and satisfies  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$ ,  $Lu = 0$  in  $\mathbb{R}_+^n$ ,  $u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}} = f$  a.e. in  $\mathbb{R}^{n-1}$ , as well as*

$$\sup_{(x', t) \in \mathbb{R}_+^n} \left[ t^{1-\eta} |(\nabla u)(x', t)| \right] \leq C \|f\|_*^{(\eta, p)}. \quad (6.1)$$

The fact that  $u$  is well-defined and is a smooth null-solution of  $L$  in the upper-half space whose nontangential trace matches  $f$  a.e. in  $\mathbb{R}^{n-1}$  follows from (2.25) with  $\varepsilon = 1$  and item (7) in Theorem 2.3. To proceed, fix an arbitrary point  $(x', t) \in \mathbb{R}_+^n$  and, making use of (3.6) and (2.17), estimate

$$|(\nabla u)(x', t)| \leq \frac{C}{t} \int_1^\infty \text{osc}_1(f; \lambda t) \frac{d\lambda}{\lambda^2} \leq \frac{C}{t^{1-\eta}} \|f\|_*^{(\eta, p)}, \quad (6.2)$$

from which (6.1) readily follows.

**Step 2.** *For every function  $u \in \mathcal{C}^1(\mathbb{R}_+^n, \mathbb{C}^M)$  there holds*

$$\|u\|_{**}^{(\eta, q)} \leq (2\eta)^{-1/2} \sup_{(x', t) \in \mathbb{R}_+^n} \left[ t^{1-\eta} |(\nabla u)(x', t)| \right]. \quad (6.3)$$

This is readily seen from (1.69).

**Step 3.** *There exists a constant  $C = C(n, L, \eta, q) \in (0, \infty)$  with the property that for every function  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$  satisfying  $Lu = 0$  in  $\mathbb{R}_+^n$  there holds*

$$\sup_{(x', t) \in \mathbb{R}_+^n} \left[ t^{1-\eta} |(\nabla u)(x', t)| \right] \leq C \|u\|_{**}^{(\eta, q)}. \quad (6.4)$$

For each fixed point  $(x', t) \in \mathbb{R}_+^n$  use Theorem 2.4 and repeated applications of Hölder's inequality in order to estimate (recall that  $Q_{x', t}$  is the cube in  $\mathbb{R}^{n-1}$  centered at  $x'$  and of side-length  $t$ )

$$\begin{aligned} |(\nabla u)(x', t)| &\leq C \int_{Q_{x', t} \times (t/2, 3t/2)} |(\nabla u)(y', s)| dy' ds \\ &= C \int_{Q_{x', t}} \left( \int_{(t/2, 3t/2)} |(\nabla u)(y', s)| ds \right) dy' \\ &\leq C \left( \int_{Q_{x', t}} \left( \int_{(t/2, 3t/2)} |(\nabla u)(y', s)|^2 ds \right)^{\frac{q}{2}} dy' \right)^{\frac{1}{q}} \\ &\leq C t^{-1/2} \left( \int_{Q_{x', t}} \left( \int_{(t/2, 3t/2)} |(\nabla u)(y', s)|^2 s ds \right)^{\frac{q}{2}} dy' \right)^{\frac{1}{q}} \\ &\leq C t^{-1} \left( \int_{Q_{x', t}} \left( \int_0^{3t/2} |(\nabla u)(y', s)|^2 s ds \right)^{\frac{q}{2}} dy' \right)^{\frac{1}{q}} \\ &\leq C t^{-1} \left( \frac{1}{|(3/2)Q_{x', t}|} \int_{(3/2)Q_{x', t}} \left( \int_0^{\ell((3/2)Q_{x', t})} |(\nabla u)(y', s)|^2 s ds \right)^{\frac{q}{2}} dy' \right)^{\frac{1}{q}} \end{aligned}$$

$$\leq Ct^{\eta-1}\|u\|_{**}^{(\eta,q)}, \quad (6.5)$$

where the last inequality is a consequence of (1.69). With this in hand, (6.4) follows.

**Step 4.** For every function  $u \in \mathcal{C}^1(\mathbb{R}_+^n, \mathbb{C}^M)$  one has

$$\sup_{\substack{x, y \in \mathbb{R}_+^n \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\eta} \leq (1 + 2/\eta) \sup_{(x', t) \in \mathbb{R}_+^n} \left[ t^{1-\eta} |(\nabla u)(x', t)| \right]. \quad (6.6)$$

In fact, the opposite inequality holds for smooth null-solutions of  $L$  in the upper-half space. Specifically, there exists a constant  $C = C(n, L, \eta) \in (0, \infty)$  with the property that for every function  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$  satisfying  $Lu = 0$  in  $\mathbb{R}_+^n$  there holds

$$\sup_{(x', t) \in \mathbb{R}_+^n} \left[ t^{1-\eta} |(\nabla u)(x', t)| \right] \leq C \sup_{\substack{x, y \in \mathbb{R}_+^n \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\eta}. \quad (6.7)$$

To justify (6.6), abbreviate

$$C_{u, \eta} := \sup_{(x', t) \in \mathbb{R}_+^n} \left[ t^{1-\eta} |(\nabla u)(x', t)| \right]. \quad (6.8)$$

Pick two arbitrary distinct points  $x = (x', t) \in \mathbb{R}_+^n$ ,  $y = (y', s) \in \mathbb{R}_+^n$ , and set  $r := |x - y| > 0$ . Then

$$r^{-\eta} |u(x) - u(y)| \leq I + II + III, \quad (6.9)$$

where

$$\begin{aligned} I &:= r^{-\eta} |u(x', t) - u(x', t + r)|, \\ II &:= r^{-\eta} |u(x', t + r) - u(y', s + r)|, \\ III &:= r^{-\eta} |u(y', s + r) - u(y', s)|. \end{aligned} \quad (6.10)$$

Then by the Fundamental Theorem of Calculus and the assumption on  $u$ ,

$$\begin{aligned} I &= r^{-\eta} |u(x', t) - u(x', t + r)| = r^{-\eta} \left| \int_0^r (\partial_n u)(x', t + \xi) d\xi \right| \\ &\leq C_{u, \eta} r^{-\eta} \int_0^r (t + \xi)^{\eta-1} d\xi \leq C_{u, \eta} r^{-\eta} \int_0^r \xi^{\eta-1} d\xi \\ &= C_{u, \eta} r^{-\eta} \eta^{-1} r^\eta = C_{u, \eta} / \eta. \end{aligned} \quad (6.11)$$

Moreover,  $III$  may be estimated in a similar manner (with the same bound  $C/\eta$ ), while

$$\begin{aligned} II &= r^{-\eta} |u(x', t + r) - u(y', s + r)| \\ &= r^{-\eta} \left| \int_0^1 \frac{d}{d\theta} [u(\theta(x', t + r) + (1 - \theta)(y', s + r))] d\theta \right| \\ &= r^{-\eta} \left| \int_0^1 (x' - y', t - s) \cdot (\nabla u)(\theta(x', t + r) + (1 - \theta)(y', s + r)) d\theta \right| \\ &\leq C_{u, \eta} r^{-\eta} |x - y| \int_0^1 [\text{dist}(\theta(x', t + r) + (1 - \theta)(y', s + r), \partial \mathbb{R}_+^n)]^{\eta-1} d\theta \\ &\leq C_{u, \eta} r^{-\eta} r \int_0^1 [(1 - \theta)s + \theta t + r]^{\eta-1} d\theta \leq C_{u, \eta} r^{-\eta} r r^{\eta-1} = C_{u, \eta}. \end{aligned} \quad (6.12)$$

Now (6.6) follows from (6.9)-(6.12).

Consider next (6.7). Recall (2.1). Fix a point  $x = (x', t) \in \mathbb{R}_+^n$  and write  $R_x$  for the cube in  $\mathbb{R}^n$  centered at  $x$  with side-length  $t/2$ . Using that the function  $u(\cdot) - u(x)$  is a null-solution of the system  $L$ , we may apply Theorem 2.4 (with  $\ell = 1$  and  $p = 1$ ) to write

$$\begin{aligned} t|(\nabla u)(x', t)| &\leq C \int_{R_x} |u(y) - u(x)| dy \\ &\leq C \|u\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}_+^n, \mathbb{C}^M)} \int_{R_x} |x - y|^\eta dy \leq C \|u\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}_+^n, \mathbb{C}^M)} t^\eta. \end{aligned} \quad (6.13)$$

This readily implies (6.7).

**Step 5.** *There exists a constant  $C = C(n, \eta) \in (0, \infty)$  such that for every continuous function  $f : \mathbb{R}^{n-1} \rightarrow \mathbb{C}^M$  one has*

$$\|f\|_*^{(\eta, p)} \leq C \sup_{\substack{x', y' \in \mathbb{R}^{n-1} \\ x' \neq y'}} \frac{|f(x') - f(y')|}{|x' - y'|^\eta}. \quad (6.14)$$

*In particular, the inclusion*

$$\dot{\mathcal{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M) \hookrightarrow \mathcal{E}^{\eta, p}(\mathbb{R}^{n-1}, \mathbb{C}^M) \quad \text{is continuous.} \quad (6.15)$$

This is a direct consequence of (1.66).

**Step 6.** *Given  $f \in \mathcal{E}^{\eta, p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ , the function  $u$  defined as in (1.25) solves the Dirichlet boundary value problem (1.70) and obeys the estimates in (1.71). Moreover,  $u \in \dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)$  and (1.72) holds as well.*

Fix an arbitrary function  $f \in \mathcal{E}^{\eta, p}(\mathbb{R}^{n-1}, \mathbb{C}^M)$ . From Step 1 we know that  $u$  given as in (1.25) is well-defined,  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$ ,  $Lu = 0$  in  $\mathbb{R}_+^n$ ,  $f = u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}}$  a.e. in  $\mathbb{R}^n$ , and satisfies (6.1). To proceed, observe that when used in concert, (6.1) and (6.3) imply that

$$\|u\|_{**}^{(\eta, q)} \leq C \|f\|_*^{(\eta, p)} \quad (6.16)$$

Hence,  $\|u\|_{**}^{(\eta, q)} < \infty$ . On the other hand, combining the results proved in Step 3 and Step 4 establishes the membership of  $u$  to  $\dot{\mathcal{C}}^\eta(\mathbb{R}_+^n, \mathbb{C}^M) = \dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)$  (cf. (2.2)) along with the estimate

$$\|u\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}_+^n, \mathbb{C}^M)} \leq C \|u\|_{**}^{(\eta, q)}. \quad (6.17)$$

Thanks to (6.16)-(6.17) and (2.2), we therefore have  $u \in \dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)$  and

$$\begin{aligned} \|f\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)} &= \|u|_{\partial \mathbb{R}_+^n}^{\text{n.t.}}\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)} = \|u|_{\partial \mathbb{R}_+^n}\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)} \\ &\leq \|u\|_{\dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)} = \|u\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}_+^n, \mathbb{C}^M)} \leq C \|u\|_{**}^{(\eta, q)} \\ &\leq C \|f\|_*^{(\eta, p)}. \end{aligned} \quad (6.18)$$

Using (6.14) and recycling part of the above estimate then yields

$$\|f\|_*^{(\eta, p)} \leq C \|f\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M)} \leq C \|u\|_{**}^{(\eta, q)}. \quad (6.19)$$

At this stage, all desired properties of  $u$  have been established.



**Step 7.** Assume that  $u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M) \cap \dot{\mathcal{C}}^\eta(\mathbb{R}_+^n, \mathbb{C}^M)$  for some  $\eta \in (0, 1)$  satisfies  $Lu = 0$  in  $\mathbb{R}_+^n$ . Then

$$u \in \dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n}, \mathbb{C}^M), \quad u|_{\partial\mathbb{R}_+^n} \in \dot{\mathcal{C}}^\eta(\mathbb{R}^{n-1}, \mathbb{C}^M) \subset L^1\left(\mathbb{R}^{n-1}, \frac{1}{1+|x'|^n} dx'\right)^M, \quad (6.20)$$

and

$$u(x', t) = \left(P_t * (u|_{\partial\mathbb{R}_+^n})\right)(x'), \quad \forall (x', t) \in \mathbb{R}_+^n. \quad (6.21)$$

To justify this, observe that the two memberships listed in (6.20) are direct consequences of (2.2) while the inclusion in (6.20) has been proved earlier (see (2.26)).

Next, fix  $x \in \mathbb{R}_+^n$ , set  $r := \text{dist}(x, \partial\mathbb{R}_+^n) > 0$ , and write interior estimates (cf. Theorem 2.4) for the function defined at each  $y \in \mathbb{R}_+^n$  by  $v(y) := u(y) - u(x)$  (which is a null-solution of  $L$ ) in the ball  $B(x, r/2) \subseteq \mathbb{R}_+^n$ . As such, we obtain

$$\begin{aligned} |(\nabla u)(x)| &= |(\nabla v)(x)| \leq \frac{C(L)}{r} \int_{B(x, r/2)} |v(y)| dy \\ &= \frac{C(L)}{r} \int_{B(x, r/2)} |u(y) - u(x)| dy \\ &\leq \|u\|_{\dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n})} \frac{C(L)}{r} \int_{B(x, r/2)} |y - x|^\eta dy \\ &\leq 2^{-\eta} C(L) \|u\|_{\dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n})} r^{\eta-1} \\ &= C(L, \eta) \|u\|_{\dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n})} \left[\text{dist}(x, \partial\mathbb{R}_+^n)\right]^{\eta-1}. \end{aligned} \quad (6.22)$$

For each fixed  $\varepsilon > 0$  consider now the function

$$u_\varepsilon(\cdot) := u(\cdot + \varepsilon e_n) \quad \text{in } \mathbb{R}_+^n. \quad (6.23)$$

Then

$$\begin{aligned} u_\varepsilon &\in \mathcal{C}^\infty(\overline{\mathbb{R}_+^n}, \mathbb{C}^M), \quad Lu_\varepsilon = 0 \quad \text{in } \mathbb{R}_+^n, \quad \text{and} \\ u_\varepsilon &\in \dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n}, \mathbb{C}^M) \quad \text{with} \quad \|u_\varepsilon\|_{\dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)} \leq \|u\|_{\dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)}. \end{aligned} \quad (6.24)$$

Since

$$\left[\text{dist}(x + \varepsilon e_n, \partial\mathbb{R}_+^n)\right]^{\eta-1} \leq \varepsilon^{\eta-1} \quad \text{for every } x \in \mathbb{R}_+^n, \quad (6.25)$$

we obtain from (6.22)-(6.24)

$$\sup_{x \in \mathbb{R}_+^n} |(\nabla u_\varepsilon)(x)| \leq C(L, \eta, \varepsilon) \|u\|_{\dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)}. \quad (6.26)$$

In light of (6.24) (which implies that  $u_\varepsilon$  is bounded on bounded subsets of  $\overline{\mathbb{R}_+^n}$ ), (6.26) allows us to conclude that

$$u_\varepsilon \in W_{\text{bd}}^{1,2}(\mathbb{R}_+^n, \mathbb{C}^M). \quad (6.27)$$

Going further, set  $f_\varepsilon(x') := u(x', \varepsilon)$  for each  $x' \in \mathbb{R}^{n-1}$ . Then, on the one hand,

$$|f_\varepsilon(x') - f_\varepsilon(y')| = |u(x', \varepsilon) - u(y', \varepsilon)| \leq \|u\|_{\dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)} |x' - y'|^\eta, \quad \forall x', y' \in \mathbb{R}^{n-1}. \quad (6.28)$$

On the other hand, for all  $x', y' \in \mathbb{R}^{n-1}$  we have (with  $\nabla'$  denoting the gradient in the first  $n-1$  variables in  $\mathbb{R}^{n-1}$ )

$$|f_\varepsilon(x') - f_\varepsilon(y')| = |u(x', \varepsilon) - u(y', \varepsilon)| \leq |x' - y'| \sup_{z' \in [x', y']} |(\nabla' u)(z', \varepsilon)|$$

$$\begin{aligned}
&= |x' - y'| \sup_{z' \in [x', y']} |(\nabla' u_{\varepsilon/2})(z', \varepsilon/2)| \\
&\leq |x' - y'| C(L, \eta)(\varepsilon/2)^{\eta-1} \|u\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}_+^n, \mathbb{C}^M)},
\end{aligned} \tag{6.29}$$

where the last inequality uses (6.22) (written for  $u_{\varepsilon/2}$  and for  $x = (z', \varepsilon/2)$ ), as well as (6.24) and (6.25) (with  $\varepsilon/2$  in place of  $\varepsilon$ ). A logarithmically convex combination of (6.28)-(6.29) then proves that for every  $\theta \in [\eta, 1]$  there exists a finite constant  $C(\theta, L, \varepsilon, u) > 0$  such that

$$|f_\varepsilon(x') - f_\varepsilon(y')| \leq C(\theta, L, \varepsilon, u) |x' - y'|^\theta, \quad \forall x', y' \in \mathbb{R}^{n-1}. \tag{6.30}$$

Hence,

$$f_\varepsilon \in \bigcap_{\eta \leq \theta < 1} \dot{\mathcal{C}}^\theta(\mathbb{R}^{n-1}, \mathbb{C}^M). \tag{6.31}$$

Combining (6.31), (6.15), and Step 6 then shows that the function

$$w_\varepsilon(x', t) := (P_t * f_\varepsilon)(x'), \quad \forall (x', t) \in \mathbb{R}_+^n \tag{6.32}$$

satisfies

$$w_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M), \quad Lw_\varepsilon = 0 \text{ in } \mathbb{R}_+^n, \quad w_\varepsilon \in \bigcap_{\eta \leq \theta < 1} \dot{\mathcal{C}}^\theta(\overline{\mathbb{R}_+^n}, \mathbb{C}^M). \tag{6.33}$$

In addition, from (6.30)-(6.32), Step 5, and Step 1, we see that  $w_\varepsilon$  has the property that for each  $\theta \in [\eta, 1)$  there exists a finite constant  $C(\theta, L, \varepsilon, u) > 0$  such that

$$\left[ \text{dist}(x, \partial \mathbb{R}_+^n) \right]^{1-\theta} |(\nabla w_\varepsilon)(x)| \leq C(\theta, L, \varepsilon, u), \quad \forall x \in \mathbb{R}_+^n. \tag{6.34}$$

In particular, choosing  $\theta \in (\max\{\eta, 1/2\}, 1)$ , the latter property allows us to estimate for every  $R > 0$

$$\begin{aligned}
\int_{B(0, R) \cap \mathbb{R}_+^n} |(\nabla w_\varepsilon)(x)|^2 dx &\leq C(\theta, L, \varepsilon, u) \int_{B(0, R) \cap \mathbb{R}_+^n} \left[ \text{dist}(x, \partial \mathbb{R}_+^n) \right]^{2(\theta-1)} dx \\
&= C(\theta, L, \varepsilon, R, u) < +\infty.
\end{aligned} \tag{6.35}$$

In concert with the last property in (6.33) (which goes to show that  $w_\varepsilon$  is bounded on bounded subsets of  $\overline{\mathbb{R}_+^n}$ ), this permits us to conclude that

$$w_\varepsilon \in W_{\text{bd}}^{1,2}(\mathbb{R}_+^n, \mathbb{C}^M). \tag{6.36}$$

From (6.24), (6.27), (6.32), (6.33), (6.36), we then conclude that the function  $v_\varepsilon := u_\varepsilon - w_\varepsilon$  belongs to  $\mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M)$  and satisfies

$$v_\varepsilon \in W_{\text{bd}}^{1,2}(\mathbb{R}_+^n, \mathbb{C}^M) \cap \dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n}, \mathbb{C}^M), \quad Lv_\varepsilon = 0 \text{ in } \mathbb{R}_+^n, \quad v_\varepsilon \Big|_{\partial \mathbb{R}_+^n} = 0. \tag{6.37}$$

Moreover, the Hölder property entails the growth estimate

$$|v_\varepsilon(x)| \leq C(1 + |x|^\eta), \quad \forall x \in \mathbb{R}_+^n, \tag{6.38}$$

where  $C := \max \{ \|v_\varepsilon\|_{\dot{\mathcal{C}}^\eta(\mathbb{R}_+^n, \mathbb{C}^M)}, |v_\varepsilon(0)| \} \in (0, \infty)$ .

The estimates near the boundary from Proposition 2.5 then imply (by sending  $\rho \rightarrow \infty$ ) that  $v_\varepsilon \equiv 0$ . This ultimately translates into saying that for each  $\varepsilon > 0$  we have

$$u(x', t + \varepsilon) = (P_t * f_\varepsilon)(x'), \quad \forall (x', t) \in \mathbb{R}_+^n. \tag{6.39}$$

Let us also note that for each  $\varepsilon > 0$ ,

$$\sup_{y' \in \mathbb{R}^{n-1}} |f_\varepsilon(y') - u(y', 0)| = \sup_{y' \in \mathbb{R}^{n-1}} |u(y', \varepsilon) - u(y', 0)| \leq \varepsilon^\eta \|u\|_{\dot{\mathcal{C}}^\eta(\overline{\mathbb{R}_+^n}, \mathbb{C}^M)}. \tag{6.40}$$

Hence,  $f_\varepsilon \rightarrow u|_{\partial\mathbb{R}_+^n}$  as  $\varepsilon \rightarrow 0^+$ , uniformly in  $\mathbb{R}^{n-1}$ . Since  $P_t$  is absolutely integrable in  $\mathbb{R}^{n-1}$ , formula (6.21) then readily follows by passing to limit  $\varepsilon \rightarrow 0^+$  in (6.39).

**Step 8.** Assume that

$$u \in \mathcal{C}^\infty(\mathbb{R}_+^n, \mathbb{C}^M), \quad Lu = 0 \quad \text{in} \quad \mathbb{R}_+^n, \quad \|u\|_{**}^{(\eta, q)} < \infty, \quad u|_{\partial\mathbb{R}_+^n}^{\text{n.t.}} = 0. \quad (6.41)$$

Then necessarily  $u \equiv 0$  in  $\mathbb{R}_+^n$ .

This is a consequence of Steps 3, 4, and 7.

**Step 9.** The end-game in the proof of Theorem 1.9.

Existence for the Dirichlet boundary value problem (1.70) follows from Step 6. Uniqueness of the Dirichlet boundary value problem (1.70) is seen from Step 8.

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